Ramanujan's sum with respect to regular integers (mod r)

Pentti Haukkanen
School of Information Sciences
FI-33014 University of Tampere, Finland

Abstract

An element a in a ring R is said to be *regular* (following von Neumann) if there exists an element $x \in R$ such that $axa = a$. If a is regular, then x is known as a pseudoinverse of a. An integer a is said to be regular (mod r) if there exists an integer $x \in \mathbb{Z}$ such that $a^2 x \equiv a \pmod{r}$. A regular integer (mod r) is regular in the ring $\mathbb{Z}_r$ in the sense of von Neumann.

An integer a is invertible (mod r) if $(a, r) = 1$. Each invertible integer (mod r) is regular (mod r). Euler's function $\phi(r)$ counts the number of invertible integers (mod r). The function $\varrho(r)$ counts the number of regular integers (mod r). The function $\varrho(r)$ is thus an analogue of Euler's ϕ-function with respect to regular integers (mod r).

The usual Ramanujan's sum $c_r(n)$ is defined as

$$c_r(n) = \sum_{\substack{a \,(\mathrm{mod}\ r) \\ (a,r)=1}} \exp(2\pi i a n / r).$$

This suggests we define Ramanujan's sum with respect to regular integers (mod r) as

$$\bar{c}_r(n) = \sum_{\substack{a \,(\mathrm{mod}\ r) \\ a \ \mathrm{regular}\,(\mathrm{mod}\ r)}} \exp(2\pi i a n / r).$$

We compare the basic properties of Ramanujan's sum and its regular analogue. We consider among others multiplicative properties, mean value and Dirichlet series.