NUMBER THEORY GROUP OULU

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Partners

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Number theory

- Primes
- Divisibility
- Diophantine equations
- Irrationality
- Continued fractions
- Cryptography

Number theory subjects of Oulu group

- Transcendental number theory
- Diophantine approximations
- Continued fractions
- Special functions
- Geometry of numbers

Irrationality and transcendence

- Irrational numbers $\mathbb{C} \setminus \mathbb{Q}$:

  Complex numbers that are not rational numbers

- Algebraic numbers $\mathbb{A}$:

  Complex numbers that are roots of some polynomials with rational coefficients.

- Transcendental numbers $\mathbb{C} \setminus \mathbb{A}$:

  Complex numbers that are not algebraic numbers.

Classical irrational numbers

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \quad \notin \mathbb{Q} \tag{1}$$

$$\log 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \quad \notin \mathbb{Q} \tag{2}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \quad \notin \mathbb{Q} \tag{3}$$

Irrational numbers

$$\prod_{n=1}^{\infty}(1+2^{-n}) \quad \notin \mathbb{Q} \tag{4}$$

$$1+\frac{1}{2+\frac{1}{3+\frac{1}{4+\cdots}}} \quad \notin \mathbb{Q} \tag{5}$$

Classical numbers whose irrationality is unknown

$$e + \pi, \quad e\pi \quad \begin{cases} \in \mathbb{Q} \\ \notin \mathbb{Q} \end{cases} \quad ? \tag{6}$$

$$\gamma = \lim_{n\to\infty}\left(\log n - \sum_{k=1}^{n}\frac{1}{k}\right) \quad \begin{cases} \in \mathbb{Q} \\ \notin \mathbb{Q} \end{cases} \quad ? \tag{7}$$

$$\zeta(5) = \sum_{n=1}^{\infty}\frac{1}{n^5} \quad \begin{cases} \in \mathbb{Q} \\ \notin \mathbb{Q} \end{cases} \quad ? \tag{8}$$

Fibology

Let F_n denote the Fibonacci numbers: $F_0 = 0, F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$.

$$\sum_{n=1}^{\infty} \frac{1}{F_n} \quad \notin \mathbb{Q} \tag{9}$$

Classical numbers/Transcendence/linear independence

- e is transcendental.
- π is transcendental.
- e^{α} is transcendental, if $\alpha \in \mathbb{A}^*$.
- $e^{\pi} = i^{-2i}$ is transcendental.

Hermite: Let $m \in \{0, 1, 2, \ldots\}$, then

$$\dim_{\mathbb{Q}}\{\mathbb{Q}e^{0} + \cdots + \mathbb{Q}e^{m}\} = m + 1. \tag{10}$$

Zeta function

Zeta function – the meromorphic continuation of the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}. \tag{11}$$

$k \in \{1, 2, \ldots\}$:

$$\zeta(k) = {}_{k+1}F_k\left(\begin{matrix} 1, \ldots, 1 \\ 2, \ldots, 2 \end{matrix} \,\middle|\, 1\right). \tag{12}$$

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^k \frac{B_{2k}(2\pi)^{2k}}{2(2k)!}. \tag{13}$$

$$\zeta(-k) = \sum_{n=1}^{\infty} n^k = -\frac{B_{k+1}}{k+1}. \tag{14}$$

Classical numbers/linear independence

Apéry, Rivoal, Ball, Zudilin:

$$\dim_{\mathbb{Q}}\{\mathbb{Q}+\mathbb{Q}\zeta(3)+\mathbb{Q}\zeta(5)+\ldots+\mathbb{Q}\zeta(2m+1)\}$$

$$= 2, \quad m = 1; \tag{15}$$

$$\geq \frac{2}{3}\frac{\log(2m+1)}{1+\log 2}. \tag{16}$$

$$\dim_{\mathbb{Q}}\{\mathbb{Q}+\mathbb{Q}\zeta(5)+\mathbb{Q}\zeta(7)+\mathbb{Q}\zeta(9)+\mathbb{Q}\zeta(11)\} \geq 2 \tag{17}$$

Irrationality measure

By an irrationality measure (or exponent) of a given number $\alpha \in \mathbb{R}$ we mean the supremum $\mu(\alpha)$ of such numbers $\mu \geq 2$ that

$$\left|\alpha - \frac{M}{N}\right| < \frac{1}{N^{\mu}} \tag{18}$$

for infinitely many $M/N \in \mathbb{Q}$.

An example of our work

Let $\mathbb{I}$ denote an imaginary quadratic field or the field $\mathbb{Q}$ of rational numbers and $\mathbb{Z}_{\mathbb{I}}$ its ring of intergers.

We show:

$$\left|\beta_0 + \beta_1 e + \beta_1 e^2 + \cdots + \beta_m e^m\right| > \frac{1}{h^{1+\epsilon(h)}}, \quad h = h_1 \cdots h_m,$$

valid for all $\overline{\beta} = (\beta_0, \ldots, \beta_m)^T \in \mathbb{Z}_{\mathbb{I}}^m \setminus \{\overline{0}\}$, $h_i = \max\{1, |\beta_i|\}$ with

$$\epsilon(h) = \frac{(4+7m)\sqrt{\log(m+1)}}{\sqrt{\log\log h}},$$

$$\log h \geq m^2(41\log(m+1)+10)e^{m^2(81\log(m+1)+20)}. \quad (19)$$

Continued fractions

$$b_0 + \operatorname*{\mathbf{K}}_{n=1}^{\infty} \frac{a_n}{b_n} = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots \tag{20}$$

$$= b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}} \tag{21}$$

Simple continued fraction

$$[b_0; b_1, b_2, \ldots] = b_0 + \operatorname*{\mathbf{K}}_{n=1}^{\infty} \frac{1}{b_n} \tag{22}$$

p-adic valuation

For any p of the set

$$\{\infty\} \cup \mathbb{P}, \quad \mathbb{P} = \{p \in \mathbb{Z}^+ \mid p \text{ is a prime}\}$$

the notation $|\ |_\infty = |\ |$ will be used for the usual absolute value of $\mathbb{C} = \mathbb{C}_\infty$ and $|\ |_p$ for the p-adic valuation of the p-adic field $\mathbb{C}_p$, the completion of the algebraic closure of the completion of $\mathbb{Q}$, defined on $\mathbb{Q}$ by

$$|q|_p = p^{-k}, \quad q = p^k \frac{a}{b}, \quad p \nmid ab.$$