Universality of multiple zeta-functions in several complex variables

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Birkhoff proved the existence of universal functions in one complex variable in 1929. However his proof was probabalistic and it was not until 1975 Voronin gave the first explicit example in his celebrated universality theorem: The Riemann zeta-function is universal. Since then many related Dirichlet series have been proved to possess the universality property. The concept of universality in several complex variables has been investigated only in recent years. Paul M. Gauthier and his students have proven general results that there exists analytic functions in n variables that are universal, similar to Birkhoff's result. However no such result is previously known for any particular Dirichlet series. The aim of our talk is to sketch a proof of the first such result, namely the multiple Hurwitz zeta-function with transcendental parameters defined by

$$\zeta(s,\alpha) = \sum_{0 \leq k_1 < k_2 < \cdots < k_n} (k_1+\alpha_1)^{-s_1} \cdots (k_n+\alpha_n)^{-s_n},$$

for $\mathrm{Re}(s_i) > 1$, and by analytic continuation elsewhere, is universal in several complex variables. This solves an open problem of Matsumoto. Our main theorem is the following:

Theorem. Let $E \subset \mathbb{C}^n$ be a Runge domain so that if $z = (z_1, \ldots, z_n) \in E$ then $1/2 < \mathrm{Re}(z_j) < 1$, where $\alpha_j > 0$ for $j = 1, \ldots, n$ are transcendental numbers and f be any holomorphic function on E. Then for any $\epsilon > 0$, and compact subset $K \subset E$ we have that

$$\liminf_{T \to \infty} \frac{1}{T^n} \operatorname{meas} \left\{ t \in [0,T]^n : \max_{z \in K} |\zeta(z+it,\alpha) - f(z)| < \epsilon \right\} > 0.$$

A simple example of a permissible set is $E = \{z \in \mathbb{C}^n : |z_i - 3/4| < \delta\}$ for any $\delta \leq 1/4$.