Inverse Scattering Theory and Transmission Eigenvalues

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Having determined the support $D$ without knowing anything about the material properties we would like to get some information about refractive index $n$.

For this we appeal to the transmission eigenvalue problem:

\[ \Delta v + k^2 v = 0 \quad \text{in} \quad D \]
\[ \Delta w + k^2 nw = 0 \quad \text{in} \quad D \]
\[ w = v \quad \text{on} \quad \partial D \]
\[ \nu \cdot \nabla w = \nu \cdot \nabla v \quad \text{on} \quad \partial D \]

**Question**: Can transmission eigenvalues be determined from scattering data?
We can use again the far field equation
\[(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z, k), \quad \text{for} \quad g \in L^2(S^2), \quad z \in D \]
and \(k \in [k_0, k_1]\).

Let \(g_\alpha := g_{z,k,\alpha}\) be the Tikhonov regularized solution of the far field equation, i.e. the solution of
\[
(\alpha I + F^* F)g = F^* \Phi_\infty(\cdot, z)
\]
Then for \(z \in D\) (except for possibly a nowhere dense set of points \(z\))
- If \(k\) is not a transmission eigenvalue \(\lim_{\alpha \to 0} \|v_{g_\alpha}\|_{L^2(D)}\) exists.
- If \(k\) is a transmission eigenvalue \(\lim_{\alpha \to 0} \|v_{g_\alpha}\|_{L^2(D)} = \infty\).
$D$ square $2 \times 2$ and $n = 16$. The far field equation is solved for several source points $z$ inside $D$ using 42 incoming directions and measurements. Red dots indicate exact eigenvalues.
Important questions in the context of inverse scattering:

- **Fredholm property of the transmission eigenvalue problem.** It arises in important questions such as uniqueness of inverse problems for inhomogenous media or justification of linear sampling methods.

- **Discreteness of transmission eigenvalues.** Methods for solving the inverse problem for inhomogeneous media such as the linear sampling method and factorization method fail at a transmission eigenvalue. Connection to uniqueness in thermo-acoustic tomography.

- **Existence of transmission eigenvalues**
  - Real transmission eigenvalues can be determined from the scattering data.
  - Transmission eigenvalues carry information about material properties.
The transmission eigenvalue problem in scattering theory was introduced by Kirsch (1986) and Colton-Monk (1988).

Research was focused on the discreteness of transmission eigenvalues for variety of scattering problems: Colton-Kirsch-Päivärinta (1989) and Rynne-Sleeman (1991).

The first proof of existence of at least one transmission eigenvalues for large contrast Päivärinta-Sylvester (2009).

The existence of an infinite set of transmission eigenvalues was first proven by Cakoni-Gintides-Haddar (2010).

Since the appearance of these papers there has been an explosion of interest in the transmission eigenvalue problem . . .

Special issue of Inverse Problems on Transmission Eigenvalues, October 2013, Cakoni-Haddar as Guest Editors
Existence of Real Transmission Eigenvalues

Theorem

Assume that $n_{\text{min}} > 1$. Then, there exists an infinite discrete set of real transmission eigenvalues $k_j$ accumulating at $+\infty$ satisfying

$$k_j(n_{\text{max}}, B_1) \leq k_j(n(x), D) \leq k_j(n_{\text{min}}, D) \leq k_j(n_{\text{min}}, B_2)$$

where $B_2 \subseteq D \subseteq B_1$.

- For $n$ constant, the first transmission eigenvalue $k_1(n)$ is strictly monotonically decreasing and continuous with respect to $n$.
- In particular, the first transmission eigenvalue uniquely determines the constant index of refraction.
- Inverse spectral problem can be solved for spherically stratified media, AKTOSUN, COLTON, GINTIDES, LEUNG, PAPANICOLAOU ...
Similar results as above can be obtained for the case when \( 0 < n_{\text{min}} \leq n(x) \leq n_{\text{max}} < 1 \).

The above analysis holds for media with voids \( D_0 \subset D \) where \( n \equiv 1 \).


The general case of sign-changing contrast \( n - 1 \) and complex valued refractive index \( n \) was first considered in


If \( n > 1 \) or \( 0 < n < 1 \) only in a neighborhood of \( \partial D \), the set of transmission eigenvalues is at most discrete.
Investigated the transmission eigenvalue spectrum for $n \in C^\infty$ such that $n \neq 1$ on the boundary $\partial D$ (could be complex valued). In particular he showed that

- there exists a discrete countable set of transmission eigenvalues
- generalized eigenfunctions span a dense subspace of the range of the resolvent.

The main idea is to employ pseudo-differential calculus to prove that a power of the resolvent is Hilbert-Schmidt.
Numerical Example: Inhomogeneous Isotropic Media

Find the constant $n_0$ with the same first transmission eigenvalue as $n(x)$. From the above $n_{\text{min}} < n(x) < n_{\text{max}}$.

<table>
<thead>
<tr>
<th>$n_e$</th>
<th>$n_i$</th>
<th>$k_1$</th>
<th>$n_0$-exact shape</th>
<th>$n_0$-recon. shape</th>
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<td>61</td>
<td>0.97</td>
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<td>59.42</td>
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</tbody>
</table>

Example from

**GIORGI-HADDAR (2012) - Inverse Problems**
Scattering by a Spherically Stratified Medium

\[ \Delta_3 u + k^2 n(r) u = 0 \quad \text{in } \mathbb{R}^3 \]

\[ u = u^i + u^s \]

\[ \lim_{r \to \infty} r \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0 \]

where \( u^i = e^{i k x \cdot d} \), \( |d| = 1 \), \( k > 0 \) is the wave number, \( r = |x| \)

and \( n(r) \) is such that \( n(r) > 0 \) and \( n(r) = 1 \) for \( r \geq a \).
Scattering by a Spherically Stratified Medium

Recall that

$$u^s(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}, d, k) + O \left( \frac{1}{r^2} \right)$$

where $\hat{x} = x/|x|$ and $\hat{x}, d \in S^2 := \{x : |x| = 1\}$.

The function $u_\infty$ is called the far field pattern of the scattered field $u^s$.

It is observed that there exist values of $k > 0$ such that

$$\int_{S^2} u_\infty(\hat{x}, d, k) ds(\hat{x}) = 0.$$ 

Such values of $k$ are transmission eigenvalues corresponding to spherically stratified eigenfunctions.

How can we analytically characterize these transmission eigenvalues?
Let $y(r)$ be the unique solution of the ODE

$$y'' + k^2 n(r)y = 0$$
$$y(0) = 0, \quad y'(0) = 1$$

Then separation of variables and elementary calculations show that $k$ is a transmission eigenvalue if and only if $y(r)$ and the solution of

$$y_0'' + k^2 y_0 = 0$$
$$y_0(0) = 0, \quad y_0'(0) = 1$$

are such that $y(a) = y_0(a), \ y'(a) = y_0'(a)$ i.e.

$$d(k) := \text{Det} \begin{vmatrix} y(a) & -\frac{\sin ka}{k} \\ y'(a) & -\cos ka \end{vmatrix} = 0.$$
$d(k)$ is an entire function of $k$ that is real for real $k$ and is bounded on the real axis. Hence if $d(k)$ is not a constant then there exist a countably infinite set of transmission eigenvalues.

**Theorem (Aktosun-Gintides-Papanicolaou)**

*If $d(k) \equiv 0$ then $n(r) \equiv 1$.***

We now assume that $n(r)$ is not identically one. If $n(a) = 1$ and $n'(a) = 0$ an asymptotic analysis shows that

$$d(k) = \frac{1}{k[n(0)]^{1/4}} \left\{ \sin k \left( a - \int_0^a \sqrt{n(\rho)} \, d\rho \right) + O \left( \frac{1}{k} \right) \right\}, \quad k \to \infty$$

and hence if $0 < n(r) < 1$ or $n(r) > 1$ there exist an infinite number of positive transmission eigenvalues. It can be shown that this is also true if $n(a) \neq 1$ and $n'(a) \neq 0$. 
Theorem (Laguerre)

Let $f(z)$ be an entire function of order less than two that is real for real $z$ and has only real zeros. Then the zeros of $f'(z)$ are also all real and are separated from each other by zeros of $f(z)$.

Examples: Laguerre’s theorem is not true in general for entire functions of order two. For example, if

$$f(z) = ze^{z^2}$$

then $f'(z) = (2z^2 + 1)e^{z^2}$ and the zeros of $f'(z)$ are complex. On the other hand if

$$f(z) = (z^2 - 4)e^{z^2/3}$$

then $f'(z) = \frac{2}{3}z(z^2 - 1)e^{z^2/3}$ and the zeros of $f'(z)$ are real but not separated by those of $f(z)$. 
Theorem (Colton-Leung)

Let \( n(r) = n_0^2 \) where \( n_0 \) is a positive constant not equal to one. Then if \( n_0 \) is an integer or the reciprocal of an integer all the transmission eigenvalues are real. If \( n_0 \) is not an integer or the reciprocal of an integer then there are infinitely many real and infinitely many complex transmission eigenvalues.

Proof (for \( n_0 > 1 \) an integer): If \( n(r) = n_0^2 \) then

\[
y(r) = \frac{1}{kn_0} \sin(kn_0r).
\]

If \( n_0 > 1 \) is an integer, the nonzero roots of \( d(k) = 0 \) are the critical points of the entire function

\[
\frac{\sin(n_0ka)}{\sin(ka)}.
\]

Hence, by Laguerre’s theorem, all zeros of \( d(k) \) are real.
[Aktosun-Gintides-Papanicolaou] Let $n(r) = n_0^2$. When $n_0 = 1/2$ we have that
\[
d(k) = -\frac{2}{k} \sin^3 \left( \frac{ka}{2} \right)
\]
and hence $d(k)$ has an infinite set of real zeros and no complex zeros. When $n_0 = 2/3$ we have that
\[
d(k) = -\frac{1}{k} \sin^3 \left( \frac{ka}{2} \right) \left[ 3 + 2 \cos \left( \frac{2ka}{3} \right) \right]
\]
and hence $d(k)$ has an infinite set of real and complex zeros.

**Question:** If complex transmission eigenvalues exist, where do they lie in the complex plane?
Let $n \in C^2[0, a]$ and let

$$
\delta := \int_0^a \sqrt{n(\rho)} \, d\rho, \quad \delta \neq a
$$

$$
A := \frac{1 + \sqrt{n(a)}}{1 - \sqrt{n(a)}}, \quad n(a) \neq 1.
$$

**Theorem (Colton-Leung)**

Let $\delta = \ell/m$, where $\ell$ and $m$ are integers, $\ell > m$ and either $|A| > 1 + \delta$ or $|A| > (\delta + 1)/(\delta - 1)$. Then there exist infinitely many real and infinitely many complex transmission eigenvalues.

**Theorem (Colton-Leung)**

Assume that $n(a) \neq 1$. Then, if complex transmission eigenvalues exist, they all lie in a strip parallel to the real axis.
Now assume that \( n(a) = 1 \) and \( n'(a) = 0 \).

1. \( d(k) \) is an even entire function of \( k \) of order (at most) one.
2. If \( 0 < n(r) < 1 \) then \( d(k) \) has a zero of order two at the origin.

Thus, by the Hadamard factorization theorem, we have that

\[
d(k) = c k^2 \prod_{j=1}^{\infty} \left( 1 - \frac{k^2}{k_j^2} \right)
\]

where \( \{k_j\} \) are the zeros of \( d(k) \) (including multiplicities) and \( c \) is a constant. From

\[
d(k) = \frac{1}{k[n(0)]^{1/4}} \left\{ \sin k \left( a - \int_0^a \sqrt{n(\rho)} \, d\rho \right) + O \left( \frac{1}{k} \right) \right\}
\]

as \( k \to \infty \) along the positive real axis we have that \( c[n(0)]^{1/4} \) is known. Hence, under the above assumptions, the transmission eigenvalues (real and complex!) determine \( [n(0)]^{1/4} d(k) \).
The Inverse Spectral Problem

As we have just seen, under appropriate assumptions the transmission eigenvalues determine \( d(k) \). In order to determine \( n(r) \) from \( d(k) \) we need an integral representation of the solution to

\[
y'' + k^2 n(r)y = 0 \\
y(0) = 0, \quad y'(0) = 1.
\]

Using the Liouville transformation

\[
\xi := \int_0^r \sqrt{n(\rho)} \, d\rho \\
z(\xi) := [n(r)]^{1/4} y(r)
\]

We arrive at

\[
z'' + [k^2 - p(\xi)]z = 0 \\
z(0) = 0, \quad z'(0) = [n(0)]^{-1/4}
\]

where

\[
p(\xi) := \frac{n''(r)}{4[n(r)]^2} - \frac{5}{16} \frac{[n'(r)]^2}{[n(r)]^3}.
\]
The solution of

\[ z'' + [k^2 - p(\xi)]z = 0 \]
\[ z(0) = 0, \quad z'(0) = [n(0)]^{-1/4} \]

can be represented in the form

\[ z(\xi) = \frac{1}{[n(0)]^{1/4}} \left[ \frac{\sin k\xi}{k} + \int_{0}^{\xi} K(\xi, t) \frac{\sin kt}{k} \, dt \right] \]

for \(0 \leq \xi \leq \delta\) where

\[ \delta = \int_{0}^{a} \sqrt{n(\rho)} \, d\rho \]

and \(K(\xi, t)\) is the unique solution of the Goursat problem

\[ K_{\xi\xi} - K_{tt} - p(\xi)K = 0, \quad 0 < t < \xi < \delta \]
\[ K(\xi, 0) = 0, \quad 0 \leq \xi \leq \delta \]
\[ K(\xi, \xi) = \frac{1}{2} \int_{0}^{\xi} p(s) \, ds, \quad 0 \leq \xi \leq \delta \]
The Inverse Spectral Problem

Theorem (Rundell-Sacks)

Let $K(\xi, t)$ satisfy the above Goursat problem. Then $p \in C^1[0, \delta]$ is uniquely determined by the Cauchy data $K(\delta, t), K_{\xi}(\delta, t)$.

Now recall the determinant

$$d(k) := \text{Det} \begin{vmatrix} y(a) & -\frac{\sin ka}{k} \\ y'(a) & -\cos ka \end{vmatrix} = 0.$$ 

From the Liouville transformation and the representation for $z(\xi)$ we have that

$$y(a) = \frac{1}{[n(0)]^{1/4}} \left[ \frac{\sin k\delta}{k} + \int_0^\delta K(\delta, t) \frac{\sin kt}{k} \, dt \right]$$

$$y'(a) = \frac{1}{[n(0)]^{1/4}} \left[ \cos k\delta + \frac{\sin k\delta}{2k} \int_0^\delta p(s) \, ds + \int_0^\delta K_{\xi}(\delta, t) \frac{\sin kt}{k} \, dt \right]$$
Note that the asymptotic formulas for \( d(k) \) gives us \( \delta \). The above formula now gives us

\[
\frac{\ell \pi}{a} d \left( \frac{\ell \pi}{a} \right) = \frac{(-1)^{\ell+1}}{[n(0)]^{1/4}} \left[ \sin \frac{\ell \pi \delta}{a} + \int_0^\delta K(\delta, t) \sin \frac{\ell \pi t}{a} \, dt \right]
\]  

(1)

and

\[
\frac{\ell \pi}{a} d \left( \frac{\ell \pi}{\delta} \right) = -y(a) \frac{\ell \pi}{\delta} \cos \frac{\ell \pi a}{\delta} + \frac{\sin \frac{\ell \pi a}{\delta}}{[n(0)]^{1/4}} \left[ (-1)^\ell + \frac{\delta}{\ell \pi} \int_0^\delta K_\xi(\delta, t) \sin \frac{\ell \pi t}{\delta} \, dt \right]
\]  

(2)
The Inverse Spectral Problem

Since \( \left\{ \sin \frac{\ell \pi t}{a} \right\} \) is complete in \( L^2[0, \delta] \) if \( \delta \leq a \) we have from (1) that \( K(\delta, t) \) (and hence \( y(a) \)) is known.

From (2) and the completeness of \( \sin \frac{\ell \pi t}{\delta} \) in \( L^2[0, \delta] \) we have that \( K_\xi(\delta, t) \) is known.

The Rundell-Sacks Theorem now implies that \( p(\xi) \) is uniquely determined for \( 0 \leq \xi \leq \delta \) from a knowledge of \( d(k) \).
From this we can now easily determine \( n(r) \).
The Inverse Spectral Problem

Theorem (Colton-Leung)

Assume that $n \in C^3[0, a]$, $n(a) = 1$ and $n'(a) = 0$. Then if $0 < n(r) < 1$ for $0 < r < a$ the transmission eigenvalues (including multiplicity) uniquely determine $n(r)$.

The only extension of the above theorem to the case of more general domains $D$ is the following:

Theorem (Cakoni-Colton-Gintides)

Let $D$ be a bounded simply connected domain with piecewise $C^1$ boundary and corresponding constant index of refraction $n$. Then $n$ is uniquely determined from a knowledge of the smallest positive transmission eigenvalue provided it is known a priori that either $n > 1$ or $0 < n < 1$. 