

Energy-band dynamics in a current-biased Josephson junction probed by incoherent Cooper-pair tunneling

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Abstract. We analyze how the dynamics in a current-biased mesoscopic Josephson junction can be probed by an additional smaller Josephson junction (probe). The probe is connected to a small island in between the Josephson junction and a resistor $R \gg R_Q = h/4e^2$. The current through the resistor results in an effective current bias to the larger JJ. We model the steady state properties of the system under thermal and quantum fluctuations of the feed current. We use a density matrix approach based on the Born-Markov equations treating the current fluctuations perturbatively. The probe is also treated perturbatively assuming the regime of incoherent Cooper-pair tunneling. We find that the $I - V$ characteristics across the probe shows traces of energy bands, Bloch oscillations and Zener tunneling in the current-biased Josephson junction.

1. Introduction

A current-biased Josephson junction (JJ) can perform Bloch oscillations in appropriate conditions [1-3]. The oscillations correspond to a periodic charging of the JJ due to the bias and discharging due to coherent Cooper-pair tunneling. If the balancing Cooper-pair tunneling does not occur, the average charge of the JJ starts to increase as the JJ makes transitions to higher bands (Zener tunneling). If Cooper-pair tunneling does not occur at later times either, at some point the quasiparticle tunneling or a current through a parallel shunt compensates the incoming current. In experiments the current-bias is provided by a nearby resistor and a voltage source in series with the JJ, which is equivalent to a parallel resistor and the current-bias. The related phenomena have been studied extensively [4] and important applications including a quantum metrological triangle [2] and a mesoscopic amplifier [5] have been proposed.

In this paper we demonstrate how to probe these effects by measuring a current through an additional smaller JJ (probe) connected to the island between the JJ and a resistor $R \gg R_Q = h/4e^2$. In this case the well known energy-band picture of the JJ is valid and quantum fluctuations of the quasicharge can be neglected (we do not model quasiparticle tunneling) [4]. The quasicharge is the variable playing the role of the quasimomentum in Bloch oscillations and corresponds to the gathered charge $\int \tilde{I} dt$ to the JJ. However, the current \tilde{I} flowing to the JJ is quantum state dependent, as different eigenstates are characterized by different voltages across the JJ. This leads to a “measurement” of the JJ’s voltage in certain timescale due to parallel (incoherent) shunt current, and is then fought back by the coherent quantum evolution due to the Josephson effect aiming to superposition of charges. Therefore for quantitative modeling also coherences of the density matrix are important in order to obtain a consistent description of the interplay between the coherent Josephson effect and the incoherent shunt current. We

model the dynamics of the system using a density matrix approach based on the Born-Markov equations treating the shunt current perturbatively. It allows us also to model the evolution, in principle, in many energy bands and for arbitrary Josephson coupling energies. Also the current across the probe is treated perturbatively using the model of incoherent tunneling [6]. The probe causes transitions between the JJ's energy bands resulting in $I - V$ characteristics that show traces of energy bands, Bloch oscillations and Zener tunneling in the current-biased Josephson junction.

2. Theory

The starting point is the Hamiltonian of the current-biased Josephson junction [1,2,4]

$$H = \frac{Q^2}{2C} - E_J \cos(\varphi) - \frac{\Phi_0}{2\pi} \varphi I, \quad (1)$$

where Q is the charge of the JJ, a conjugated variable to the phase-difference φ , C its capacitance, E_J the Josephson coupling energy and I the bias current. We represent the density matrix in the eigenbasis of $H_0 = Q^2/2C - E_J \cos(\varphi)$ which has an energy band structure $E^n(q)$ as a function of the quasicharge q , or the quasimomentum $k = q/2e$, and the band index n . The quasicharge is the charge fed to the JJ by the external circuit and does not exist in superpositions ($\alpha = R_Q/R \ll 1$). Therefore the components of the density matrix $\rho = \sum_{nn'} \int dk \int dk' \rho^{nn'}(k, k') |n, k\rangle \langle n', k'|$ take the form $\rho^{nn'}(k, k') = \rho^{nn'}(k) \delta(k - k')$. In the following, for simplicity, we group the density matrix equation of motion due to the feed current to similar terms as $\dot{\rho} = \dot{\rho}_L + \dot{\rho}_{\delta I} + \dot{\rho}_T + \dot{\rho}_{\delta I+T}$.

2.1. The Liouville equation of motion

The Hamiltonian (1) leads to the Liouville equation of motion $\dot{\rho} = \frac{i}{\hbar} [H, \rho] = \frac{i}{\hbar} [H_0, \rho] + \frac{i}{\hbar} [H_I, \rho]$, where $H_I = -\Phi_0 \varphi I / 2\pi$. The commutator with H_0 produces oscillations of the superpositions as we work in the eigenbasis of H_0 . For commuting with H_I one can use the expression $\varphi_{kk'}^{nn'} = i\delta_{nn'} \partial \delta(k - k') / \partial k + \varphi_k^{nn'} \delta_{nn'} \delta(k - k')$ [2], leading to the equation of motion

$$\dot{\rho}_L^{nn'}(k) = -i\omega_k^{nn'} \rho^{nn'}(k) - \frac{I}{2e} \frac{\partial \rho^{nn'}(k)}{\partial k} + i\frac{I}{2e} \left[\sum_{n_i \neq n} \varphi_k^{nn_i} \rho^{n_i n'}(k) - \sum_{n'_i \neq n'} \varphi_k^{n'_i n'} \rho^{nn'_i}(k) \right], \quad (2)$$

where $\omega_k^{nn'} = [E^n(k) - E^{n'}(k)]/\hbar$. The first term in the r. h. s. of Eq. (2) corresponds to the oscillations of the superpositions, the second one is due to the current-bias and results in a movement $\dot{q} = I$, and the last one describes interband (Zener) tunneling due to the same term.

2.2. Including the current noise

The fluctuations of the feed current due to the shunt are taken into account as a perturbation, producing both classical (thermal) and quantum noise to the system. We add to the constant current I a fluctuating component \hat{I} satisfying the quantum mechanical equilibrium correlations

$$D(\omega) = \text{Re} \left\{ \left(\frac{\Phi_0}{2\pi} \right)^2 \lim_{s \rightarrow 0} \int_0^\infty dt \langle \hat{I}(t) \hat{I}(0) \rangle e^{i\omega t - st} \right\} = \frac{\hbar^2}{2\pi} \frac{R_Q}{R} \frac{\omega}{1 - e^{-\hbar\omega/k_B T}}, \quad (3)$$

where the restriction to the real part is justified by the assumption $\alpha \ll 1$ (causing the diagonality in q). The noise is accounted for by additional terms in the density matrix equation of motion. The Born-Markov equations [7] result in three kind of contributions, the first one describing quasicharge dynamics with no interband tunneling [4]

$$\dot{\rho}_{\delta I}^{nn'}(k) = \frac{\alpha k_B T}{2\pi \hbar} \frac{\partial^2 \rho^{nn'}(k)}{\partial k^2} + \frac{\alpha}{4\pi \hbar} \frac{\partial}{\partial k} \left\{ \rho^{nn'}(k) \left[\frac{\partial E^n(k)}{\partial k} + \frac{\partial E^{n'}(k)}{\partial k} \right] \right\}, \quad (4)$$

the second one describing interband transitions with no quasicharge dynamics

$$\begin{aligned}
\dot{\rho}_T^{nn'}(k) &= \frac{1}{\hbar^2} \sum_{n_i \neq n, n'_i \neq n'} \rho^{n_i n'_i}(k) \varphi_k^{nn_i} \varphi_k^{n'_i n'} (D_k^{n_i n} + D_k^{n'_i n'}) \\
&- \frac{1}{\hbar^2} \sum_{n_i, n_v (n_i \neq n_v \neq n)} \rho^{n_i, n'}(k) \varphi_k^{n_v n_i} \varphi_k^{nn_v} D_k^{n_i n_v} \\
&- \frac{1}{\hbar^2} \sum_{n'_i, n_v (n'_i \neq n_v \neq n')} \rho^{n, n'_i}(k) \varphi_k^{n_v n'} \varphi_k^{n'_i n_v} D_k^{n'_i n_v}, \tag{5}
\end{aligned}$$

where $D_k^{nn'} = D(\omega_k^{nn'})$, and the third one describing the mixing of both the processes

$$\begin{aligned}
\dot{\rho}_{\delta I+T}^{nn'}(k) &= -\frac{i}{\hbar^2} \frac{\partial}{\partial k} \sum_{n_i \neq n} \rho^{n_i, n'}(k) D_k^{n_i n} \varphi_k^{nn_i} + \frac{i}{\hbar^2} \frac{\partial}{\partial k} \sum_{n'_i \neq n'} \rho^{n, n'_i}(k) D_k^{n'_i n'} \varphi_k^{n'_i n'} \\
&+ i \sum_{n'_i \neq n'} \varphi_k^{n'_i n'} \left\{ \frac{\alpha k_B T}{2\pi\hbar} \frac{\partial \rho^{nn'_i}(k)}{\partial k} + \frac{\alpha}{4\pi\hbar} \rho^{nn'_i}(k) \left[\frac{\partial E^{n'_i}(k)}{\partial k} + \frac{\partial E^{n_f}(k)}{\partial k} \right] \right\} \\
&- i \sum_{n_i \neq n} \varphi_k^{nn_i} \left\{ \frac{\alpha k_B T}{2\pi\hbar} \frac{\partial \rho^{n_i n'}(k)}{\partial k} + \frac{\alpha}{4\pi\hbar} \rho^{n_i n'}(k) \left[\frac{\partial E^{n'}(k)}{\partial k} + \frac{\partial E^{n_i}(k)}{\partial k} \right] \right\}. \tag{6}
\end{aligned}$$

2.3. Incoherent Cooper-pair tunneling across the probe

Assuming that the Cooper-pair tunneling across the probe is weak (if compared, for example, to the shunt current), it can be treated perturbatively as incoherent [6]. The tunneling occurs mainly nearby degenerate situations, where the energy $2eV_p$ released in a tunneling of a Cooper pair, V_p being the bias voltage of the probe (see figure 1), matches to an energy level difference of a populated and an arbitrary eigenstate. This results in a Lorentzian type (forward or backward) tunneling rate between diagonal elements of the density matrix

$$\Gamma_{fi}^{\pm} = \frac{E_p^2}{\hbar} |\langle f | e^{\mp i\varphi} | i \rangle|^2 \frac{\Delta}{4(E_f - E_i \pm 2eV_p)^2 + \Delta^2}, \tag{7}$$

where E_p is the Josephson coupling energy of the probe and Δ the broadening of the resonances. The matrix elements are calculated in the basis of fixed quasicharge [4,6]. The corresponding decay terms in the density matrix equation are then $\dot{\rho}^{ij} = -\rho^{ij} \sum_f (\Gamma_{fi}^+ + \Gamma_{fi}^- + \Gamma_{fj}^+ + \Gamma_{fj}^-)/2$.

2.4. Discretizing the quasicharge and calculating the steady state properties

For numerical simulations the quasicharge and derivatives with respect to it have to be discretized. We do this in a symmetric manner $\partial\rho/\partial k \rightarrow n(\rho_{k+1/n} - \rho_{k-1/n})/2$ and $\partial^2\rho/\partial k^2 \rightarrow n(\rho_{k+1/n} - 2\rho_k + \rho_{k-1/n})$, where n is the number of points in the discretized quasimomentum space $0, 1/n, 2/n \dots 1 - 1/n$ and ρ_k the value $\rho(k)$. The derivatives of the energies have to be discretized carefully, in order to balance transition and decay rates of the quasicharge states. For example, the definitions $\partial E/\partial k \rightarrow n(E_{k+1/n} - E_k)$ and $\partial^2 E/(\partial k^2) \rightarrow n(E_{k+2/n} - E_{k+1/n} - E_k + E_{k-1/n})$ lead to this property and preserves the trace of the density matrix. As our equation is not of Lindblad type, the positiveness of the density matrix is not guaranteed either, but according to our numerical simulations it holds for dense enough discretation. For calculating the average properties of the system, we deduce the steady state of the density matrix by Laplace transforming the equations of motion and using the limit $s \rightarrow 0$. The average voltage V across the JJ is $\langle Q \rangle / C$ and the the current across the probe is obtained from the transition rates (7).

3. Numerical results

If $E_p = 0$ (no probe) the model reproduces the results of Refs. [2,4]. For small I the quasicharge fluctuates, due to thermal noise characterized by $k_B T R_Q / R$, in the lowest band nearby the point which leads to a shunt current I . For larger currents Bloch oscillations become possible and the distribution extends to the whole band. For high currents Zener tunneling occurs until the JJ is in a band which can cause high enough voltage to compensate the bias current I . Also incoherent Cooper-pair tunneling across the larger JJ emerges and induces downward transitions.

If $E_p \neq 0$, the dynamics are modified by new transitions between energy bands. Again, for low I the distribution settles nearby a point in the lowest band, except when $2eV_p$ matches to an energy level difference to a higher band (at this point). Then Cooper-pairs tunnel across the probe exciting the JJ. The tunneling can occur to both directions (different signs of V_p) but for $E_J < E_C$ one of the directions dominates, because the eigenstates are close to the charge states. After the tunneling, the quasicharge travels back to its original position by “sliding” down the excited bands and making downward transitions nearby the Zener tunneling regions. As I is varied, the mean position of the quasicharge, and also the resonant voltage, is changed (figure 2). For larger I Bloch oscillations emerge and the excitation can occur at any point of the band for a fixed I . The probe current is enhanced for a wide range of voltages (figure 2). If $E_C > E_J$, higher I causes Zener tunneling and a narrow distribution in one of the excited bands, which is detected by a similar but opposite direction tunneling (transition to the ground band). The resonances are seen for opposite values of V_p (figure 2). For $E_J \gg E_C$ interband tunneling is weak for reasonable I but can be induced by the probe.

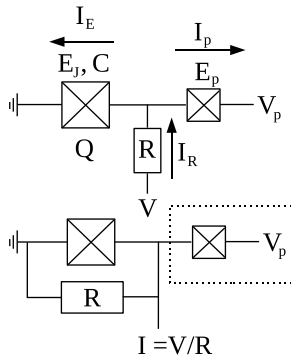


Figure 1. The schematic diagram of the circuit modeled (upper diagram) and the one used in the simulation (lower). The crossed boxes represent JJs and the dotted region is treated perturbatively.

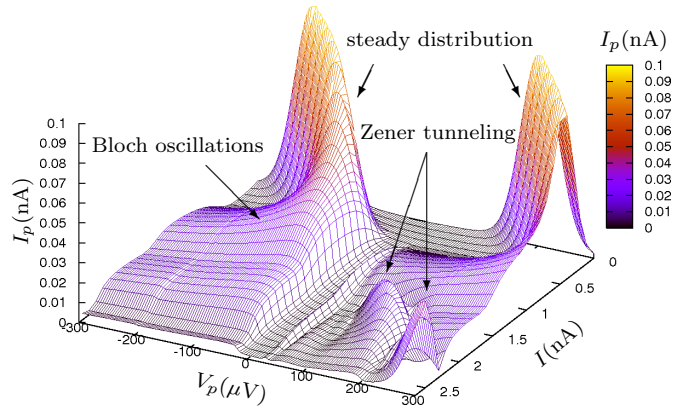


Figure 2. Numerical results for the absolute value of the probe current as a function of V_p and I . The simulation is done for three lowest energy bands using a quasicharge discretation $n = 200$. The parameters are $E_C = 2E_J = 20E_p = 6.5\Delta = 100 \mu\text{eV}$, $T = 50 \text{ mK}$ and $R = 200 \text{ k}\Omega$.

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