Mathematical preliminaries

1. **Plane polar basis vectors**
   For plane or cylindrical polar coordinates $\hat{\boldsymbol{r}} = \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta$ and $\hat{\boldsymbol{\theta}} = -\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta$, see appendix B of the lectures. Express $\boldsymbol{i}$, $\boldsymbol{j}$ in terms of $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$.

2. **Taylor's theorem for a function of several variables**
   Generalise the one dimensional Taylor's theorem for three dimensions $\phi(x_1 + h_1, x_2 + h_2, x_3 + h_3)$ by considering all the coordinates separately and ending at second degree terms in $\boldsymbol{h}$. Show that it may be put as
   $$\phi(\boldsymbol{x} + \boldsymbol{h}) = \phi(\boldsymbol{x}) + \boldsymbol{h} \cdot (\boldsymbol{\nabla}\phi)_{\boldsymbol{x}} + O(\boldsymbol{h}^2)$$
   or as
   $$\phi(\boldsymbol{x} + \boldsymbol{h}) = \phi(\boldsymbol{x}) + h_j(\partial\phi/\partial x_j)_{\boldsymbol{x}} + O(h_k h_k).$$

3. **Derivatives of curvilinear basis vectors**
   For spherical polar coordinates calculate $\partial\hat{\boldsymbol{r}}/\partial\theta$, $\partial\hat{\boldsymbol{\theta}}/\partial\theta$, $\partial\hat{\boldsymbol{\theta}}/\partial\lambda$, $\partial\hat{\boldsymbol{\lambda}}/\partial\lambda$; in each case express your answer in terms of the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\lambda}}$ and not in terms of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$.

4. **Vector derivative formulas**
   Show that
   (a) $\boldsymbol{\nabla} \times (\partial\boldsymbol{A}/\partial t) = \partial/\partial t(\boldsymbol{\nabla} \times \boldsymbol{A})$
   (b) $\boldsymbol{\nabla} \cdot (\phi\boldsymbol{\nabla}\psi) = \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi + \phi\nabla^2\psi$
   (c) $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$
   (d) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = 0$
   (e) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A}$.
   Hint: Try to use the index notation shown in the appendix A of the lecture notes.

5. **Estimation of the mean free path in air**
   An estimate for the mean free path λ of gas particles can be based on the equation
   $$\lambda \approx \frac{1}{n\sigma}, \tag{1}$$
   where $n = N/V$ is the number density of particles and σ is the scattering cross section. Estimate λ in standard temperature and pressure ($T = 273$ K and $p = 10^5$ Pa) using this formula and $\sigma \approx \pi a^2$, where $a = 150$ pm is the bond length of N_2 molecule. Use the equation of state of an ideal gas $pV = Nk_BT$, where the Boltzmann's constant $k_B = 1.381 \cdot 10^{-23}$ J K^{-1}.

   [Note that equation (**??**) can be derived as follows: when a particle of cross section σ travels distance λ, it sweeps volume $V = \sigma\lambda$. For one collision to occur in this distance, we must have approximately one particle in this volume, $nV \approx 1$.]

1. **Green's identity**
   Show that

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV = \int_S (\phi \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \phi) \cdot\, d\boldsymbol{S}.$$

2. **3 D flow**
   Calculate and describe particle paths and streamlines for the flow

$$\boldsymbol{v} = (ay, -ax, b(t)).$$

   What could be modelled by the case $b(t) =$ constant?

3. **Streamlines**
   Sketch streamlines for

$$\begin{aligned} &(a) \quad \boldsymbol{v} = (a \cos \omega t, a \sin \omega t, 0), \\ &(b) \quad \boldsymbol{v} = (x - Vt, y, 0), \\ &(c) \quad v_r = r \cos \frac{\theta}{2},\ v_\theta = r \sin \frac{\theta}{2},\ v_z = 0,\ 0 < \theta < 2\pi. \end{aligned}$$

4. **Streamlines and particle paths**
   Find streamlines and particle paths for the two-dimensional flows

$$\begin{aligned} &(a) \quad \boldsymbol{v} = (xt, -yt, 0), \\ &(b) \quad \boldsymbol{v} = (xt, -y, 0). \end{aligned}$$

1. **Stream function**
Verify that the example flow $\boldsymbol{v} = (ax, -ay, 0)$ satisfies the continuity equation with constant a and constant density. Determine the stream function ψ. Discuss the particle paths based on ψ.

2. **Radial flow**
   (a) Using the method explained in the book or in the appendix of the lecture notes, calculate $\boldsymbol{\nabla} \cdot (f(r)\hat{\boldsymbol{r}})$ in a spherical system of coordinates.
   (b) Let $\boldsymbol{v} = mr^{-2}\hat{\boldsymbol{r}}$ in a spherical system of coordinates. Show that $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ except at origin O. Let S be *any* smooth surface surrounding O. Show that volume flows through S at rate $4\pi m$. What is the corresponding result if O lies on S?

3. **Circular and pipe flow**
   (a) Calculate $D\boldsymbol{v}/Dt$ for the steady two-dimensional circular flow $\boldsymbol{v} = f(r)\hat{\boldsymbol{\theta}}$. Does your result fit in with particle dynamics?
   (b) Water flows along a pipe whose area of cross-section $A(x)$ varies slowly with the coordinate x along the pipe. Express the mass flow at x using $A(x)$, the density ρ and the velocity $\boldsymbol{v}_{\text{ave}}(x) \approx v_{\text{ave}}\boldsymbol{i}$, which is averaged over the cross section of the pipe. Use the conservation of mass to determine $\boldsymbol{v}_{\text{ave}}(x)$ in the pipe, and calculate the acceleration of a particle moving with this averaged velocity.

4. **Flow around a cylinder**
A flow around a cylinder can be described by the stream function

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta,$$

where U is a constant and a denotes the radius of the cylinder.
   (a) Show that there is no flow through the surface $r = a$ of the cylinder.
   (b) Calculate the tangential velocity v_θ on the surface of the cylinder.
   (c) Find the stream lines corresponding to $\psi = naU$ (n integer) by calculating their positions when $x \to \infty$ and at $x = 0$, and sketching the rest.

1. **Solution of Laplace equation by separation of variables**
by For applications later in this course, go through the solution of the Laplace equation given in the appendix C of the lecture notes. Find $\phi(x, y)$ for the cases
   a) $f(y) = C\delta(y - \frac{a}{2})$,
   b) $f(y) = C \sin \frac{\pi y}{a}$.

   Hint: Function $\delta(y - \frac{a}{2})$ denotes Dirac delta (δ) function at $y = \frac{a}{2}$. Generally, δ function is defined with help of integration:

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\mathrm{d}x = f(x_0).$$

2. **Vorticity and deformation**
Given the flow

$$\boldsymbol{v} = (3z + 4x, -5y, -2x + z),$$

   calculate the vorticity and the symmetric and antisymmetric parts of $\partial v_i/\partial x_j$.

3. **Vorticity and deformation in Poiseuille flow**
Poiseuille flow in a pipe has velocity components

$$u = v = 0, \ w = b(a^2 - x^2 - y^2),$$

   where $\boldsymbol{v} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$.
   a) Calculate $\boldsymbol{\nabla} \cdot \boldsymbol{v}$ and $\boldsymbol{\nabla} \times \boldsymbol{v}$.
   b) Calculate the symmetric and antisymmetric parts of $\partial v_i/\partial x_j$.
   c) Find the eigenvalues and eigenvectors (principal axes) of the symmetric part.
   d) Express the vorticity in cylindrical polar coordinates and discuss the direction of the vorticity in terms of the slipping of layers of fluid over each other.

4. **Vortex**
A vortex has the stream function $\psi = -C \ln \frac{r}{a}$. Calculate the vorticity outside of the line ($r = 0$) to show that $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$. Show, by using the Stokes' theorem, that the circulation $\kappa = \oint \boldsymbol{v} \cdot d\boldsymbol{l}$ for vortex flow is the same for any simple curve once around the origin (in the positive direction).

1. **Accelerated frame**
   A frame of reference which is accelerating (with respect to inertial frames) is used to describe an experiment. The acceleration has the constant value f in the $\boldsymbol{i}$ direction.
   (a) Show that an 'inertial force' $-\rho f \boldsymbol{i}$ acts on a fluid (per volume) and that a potential fx may be used.
   (b) Hence find the equilibrium water surface in an accelerated tank of water by taking into account also the gravity.

   Hint: It will be shown in the next lecture that the liquid surface corresponds to potential $\Phi =$ constant.

2. **Area projection**
   Show that the relation $\delta A_1 = \boldsymbol{n} \cdot \boldsymbol{i} \delta A$ used in the lectures is valid.

   Hint: Express $\boldsymbol{n}\delta A$ as a cross product of two vectors.

3. **Normal and tangential stress forces**
   Show that the normal component of the surface force vector is

$$\sigma_{ij} n_j n_i$$

   per area, and find an expression for the tangential force on area dS (i.e. the force parallel to the surface).

4. **Hydrostatics of Earth's rotation**
   Let us assume that the Earth's gravitational field is isotropic.
   (a) Show that due to Earth's rotation the ocean surface varies as

$$\delta r = \frac{\Omega^2 R^2 \sin^2 \theta}{2g}, \tag{2}$$

   where R is the radius and Ω is the angular velocity of Earth and θ is the polar angle.

   (b) How could you justify the use of constant radius R in the right hand side of equation (**??**)?

   (c) Compare the results with measured values

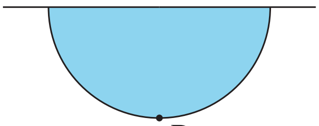
$$\begin{aligned} R_{\text{pole}} &= 6\,356\,912\,\text{m}, \\ R_{\text{equator}} &= 6\,378\,388\,\text{m}. \end{aligned}$$

   What could cause the difference?

   Hint: Liquid surface corresponds to $\Phi =$ constant.

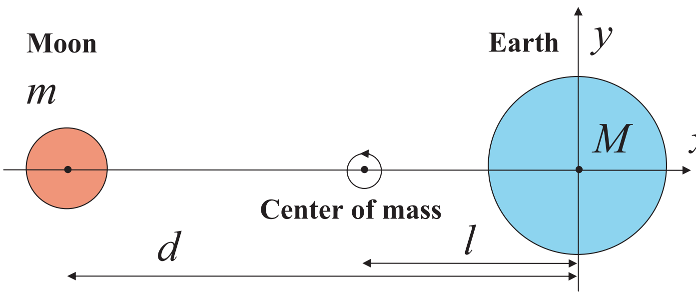
1. **Integration formulas**
   Derive the formulae: a) $\int_S f d\boldsymbol{S} = \int_V \boldsymbol{\nabla} f dV$, and b) $\int_S A_{ij} dS_j = \int_V \frac{\partial A_{ij}}{\partial x_j} dV$. [Hint: Multiply by a constant vector and use the divergence theorem.]

2. **Hydrostatic forces**
   A gutter is in the form of half a cylinder and is full of water (see figure).
   a) Prove, by integrating surface forces, that the total force on the gutter is equal to the weight of water in the gutter.



   b) Calculate the moment, about the lowest level of the gutter, of the surface forces on the half of the gutter on one side of this lowest line.
   c) Calculate the force on one half of the gutter.

3. **Hydrostatic theory of tides** (solving this problem gives double points)
   The starting point for studying tides is to consider the Earth and the Moon circulating around their center of mass with angular velocity Ω. Tides are caused by the effect of the Moon's gravitational potential $\Phi_m = -\gamma m / r'$ (where r' is the distance from Moon's center) on the surface of the Earth.



M and m are the masses of the Earth and the Moon, respectively, and d is the distance between the centers of the Earth and the Moon. The rotation axis of the Earth-Moon system is perpendicular to the $x - y$ plane.

   a) Determine the distance l of the center of mass from the center of the Earth.
   b) Express the potential Φ_m as a function of x, y and z.
   c) By expanding Φ_m in Taylor series up to second order in x/d, y/d and z/d, and neglecting all constant and higher-order terms show that

$$\Phi_m = \frac{\gamma m}{d^2} x - \frac{\gamma m}{2d^3}(2x^2 - y^2 - z^2) \tag{1}$$

   d) We now argue that the term linear in x in Φ_m (1) causes the centripetal acceleration that keeps the Earth at constant distance from the center of mass. Show that this leads to the condition $\Omega^2 d^3 = \gamma(m + M)$.
   e) Take into account also Earth's gravitational potential near the surface $\Phi_e = gh$. (Here h is the height and the g can also be expressed as $g = \gamma M / R_e^2$, where R_e is the radius of the Earth.) Using this together with the quadratic terms in (1),

express the condition for the see level in hydrostatic equilibrium, and calculate numerically the maximum height of the tide. (Warning: assuming hydrostatic equilibrium severely underestimates the tide near coastlines. Also other bodies, especially the Sun, contribute to tides.)

1. **Microscopic model of viscosity**
The viscosity of a gas can be estimated as $\mu = \frac{1}{3}\rho v \lambda$, where $v = \sqrt{3k_b T/m}$ is the average velocity of molecules and λ is the mean free path. Estimate μ numerically for air (use the results $\lambda = 570$ nm from exercise set 1, problem 5, and mass $m = 4.65 \cdot 10^{-26}$ kg for N_2) and compare with the measured value.

2. **Viscous stress tensor** (solving this problem gives double points)
(a) The form of the stress tensor, assuming $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$,

$$\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

is valid in cartesian coordinates. Applying the formulas given in appendix B of the lecture notes show that in plane polar coordinates the stress tensor takes the form

$$\sigma_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r}, \qquad \sigma_{\theta r} = \sigma_{r\theta} = \mu \left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right),$$

$$\sigma_{\theta\theta} = -p + 2\mu \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right).$$

(b) The stream function

$$\Psi = U \left(r - \frac{a^2}{r} \right) \sin\theta$$

gives a model flow past a cylinder of radius a. Calculate the components of the viscous stress tensor σ'_{ij} in plane polar coordinates.
(c) Calculate the total viscous force on the cylinder. Is it realistic?

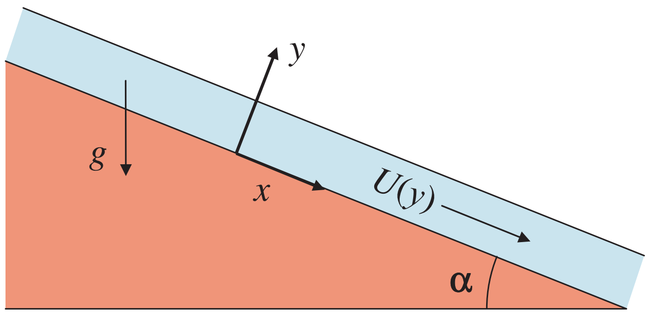
3. **Reynold's number estimates**
Calculate the Reynolds number and comment the relative importance of inertial and viscous forces in the following cases:
   a) A swimmer's kick: $a = 50$ cm, $v = 30$ cm/s.
   b) A bacterium in water: $a = 1$ μm, $v = 30$ μm/s.
   c) A river: $a = 10$ m, $v = 10$ cm/s.
   d) The climate: $a = 1000$ km, $v = 10$ m/s.
   e) A glacier: $a = 100$ m, $v = 1$ m/year, $\mu \sim 10^{10}$ kg/(m s).
   f) An accretion disk around a black hole:
   $a = 10^5$ m, $v = 10^7$ m/s and $\nu \sim 10^2$ m^2/s.

1. **Plane Couette flow**
   Consider fluid between parallel planes. The wall at $y = 0$ is fixed, and the wall at $y = a$ moves with steady speed V in its own plane. Solve the Navier-Stokes equations for the case $\rho =$ constant to show that a possible flow is

$$\boldsymbol{v} = \frac{Vy}{a}\boldsymbol{i}.$$

   Calculate the forces on both walls.

2. **Flow down a slope** (solving this problem gives double points)
   A liquid of constant density flows down a plane which slopes at angle α to the horizontal, as indicated in the figure below. The free surface of the liquid is at a uniform distance from the plane, has pressure p_0 and no shear stress. For this flow you need to keep the gravitational field in the Navier-Stokes equation, as it is now dynamically active. Set up and solve equations for $U(y)$, and verify that the forces on a length l of the fluid layer are in equilibrium.

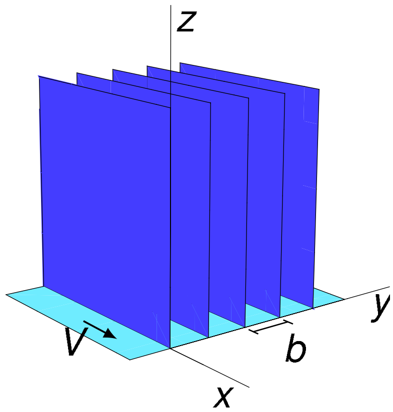


1. **Dimensioless Euler's equation**
   The Euler equation for incompressible flow in a rotating system was given in the form

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + 2\rho \boldsymbol{\Omega} \times \boldsymbol{v} = -\boldsymbol{\nabla} p$$

   in the lectures. Write this in a dimensionless form.

2. **Paintbrush** (solving this problem gives double points)
   Consider a simple model of a paintbrush consisting of parallel planes with spacing b and normal $\boldsymbol{j}$.



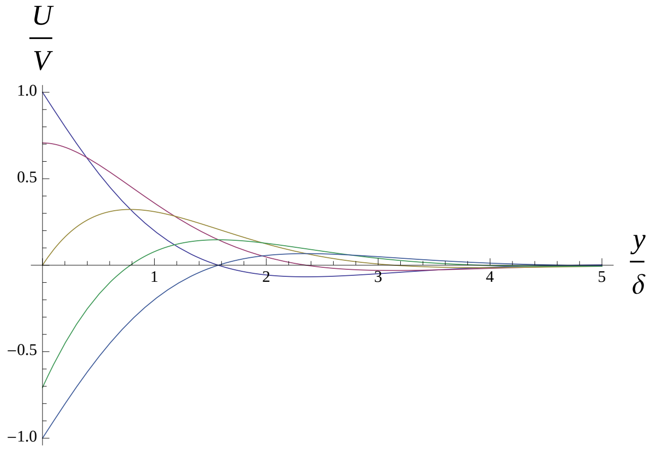
   For convenience, assume the wall in the $x - y$ plane is moving with constant velocity $V\boldsymbol{i}$ and that the brush is stationary.
   Determine the velocity of the paint between the brush planes assuming the form $\boldsymbol{v} = U(y, z)\boldsymbol{i}$. (Use the method of separation of variables.)
   Calculate the total paint flow $Q = \int_0^b dy \int_0^\infty dz\, U(y, z)$ between two planes. Based on this deduce how thick is the layer of paint left on the wall.
   [Answer: $Q = \frac{8Vb^2}{\pi^3} \sum_{n=1}^\infty (2n-1)^{-3} \approx 0.27Vb^2$.]

3. **Oscillating plane**
   The plane $y = 0$ oscillates transversally with velocity $\boldsymbol{i}V\cos(\omega t)$. Show that the velocity of fluid $\boldsymbol{v} = U(y, t)\boldsymbol{i}$ above the plane ($y > 0$) has the form

$$U(y, t) = \Re[Ve^{i\omega t - (1+i)y/\delta}],$$

   where $\delta = \sqrt{2\nu/\omega}$, i is the imaginary unit ($i^2 = -1$) and $\Re$ means the real part. Calculate the real part and discuss its form. Why is δ called "penetration depth"?

   [Hint: Use ansatz $U = Ve^{i\omega t - ky}$ to solve the Navier-Stokes equations. This gives a complex solution but the real part of this corresponds to the physical solution.]



The figure shows $U(y,t)$ plotted at five different time instances.

1. **Transient flow between parallel planes** (double points)
   Fluid is at rest in a long channel with rigid walls $y = \pm a$ when a pressure gradient $-G$ is suddenly imposed at $t = 0$.
   a) Show that the velocity $U(y,t)\boldsymbol{i}$ satisfies the equation

   $$\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial y^2} + \frac{G}{\rho}$$

   for $t > 0$, and state the boundary and initial conditions for this flow.
   b) As $t \to \infty$ we expect to get the flow appropriate for a pressure gradient in a channel

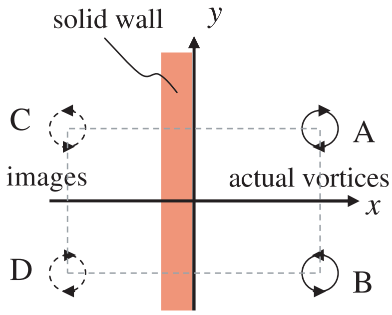
   $$U_1(y) = \frac{G}{2\mu}(a^2 - y^2)$$

   so seek a solution in the form

   $$U(y,t) = U_1(y) + V(y,t) :$$

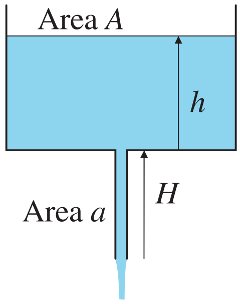
   what equation and boundary values does V satisfy?
   c) Show that $V(y,t)$ may be found by separation of variables. How long does it take for the flow U_1 to be established? Explain this answer physically.

2. **Vortex pair near a wall**
   Consider a pair of vortices, A and B, of circulations $-\kappa$ and κ, respectively, approaching a wall. The boundary condition for the normal component of the velocity at the wall, $v_x(0,y) = 0$, can be satisfied by adding two "image vortices" C and D, with circulations κ and $-\kappa$, respectively, behind the wall.

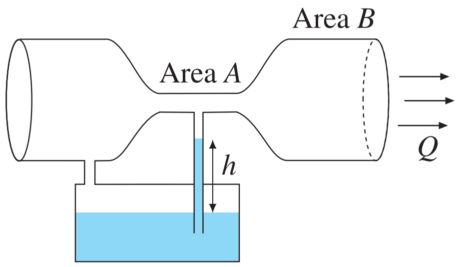


   a) Calculate the velocity at A induced by vortices B, C and D.
   b) Formulate a differential equation for the path of vortex A.
   c) Show that its solution is $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{x_0^2} + \frac{1}{y_0^2}$, and sketch the trajectory.

1. **Discharge from a container with a drain pipe**
   Water flows out of the reservoir down a pipe of cross-sectional area a, see the figure below. What is the speed of the issuing jet of water? Estimate the time to empty the reservoir. Describe what happens if there is a small hole in the pipe half way down.



2. **Venturi flow**
   Air is drawn at volume rate Q along a horizontal pipe through a contraction. The pipe is connected to a water tank as sketched in the figure below. Estimate the height h for which water can be sucked into the vertical pipe attached at the constriction.



3. **Train in a tunnel**
   A train travels at speed 150 km/h in a tunnel. How is the air pressure inside the train modified compared to the case that the train would be stationary. Assume the dimensions of the train are width 3 m, height 4 m and length 100 m, and the corresponding dimensions of the tunnel are 5 m, 7 m and 2 km. (Hint: do all possible simplifying assumptions so that you still get a non-vanishing result.)

TURN

4. **Free fall**

A falling object in a medium reaches a terminal velocity where the gravitational force and the drag force of the medium balance each other. Using the attached graph, estimate the terminal velocities of the following objects in air:

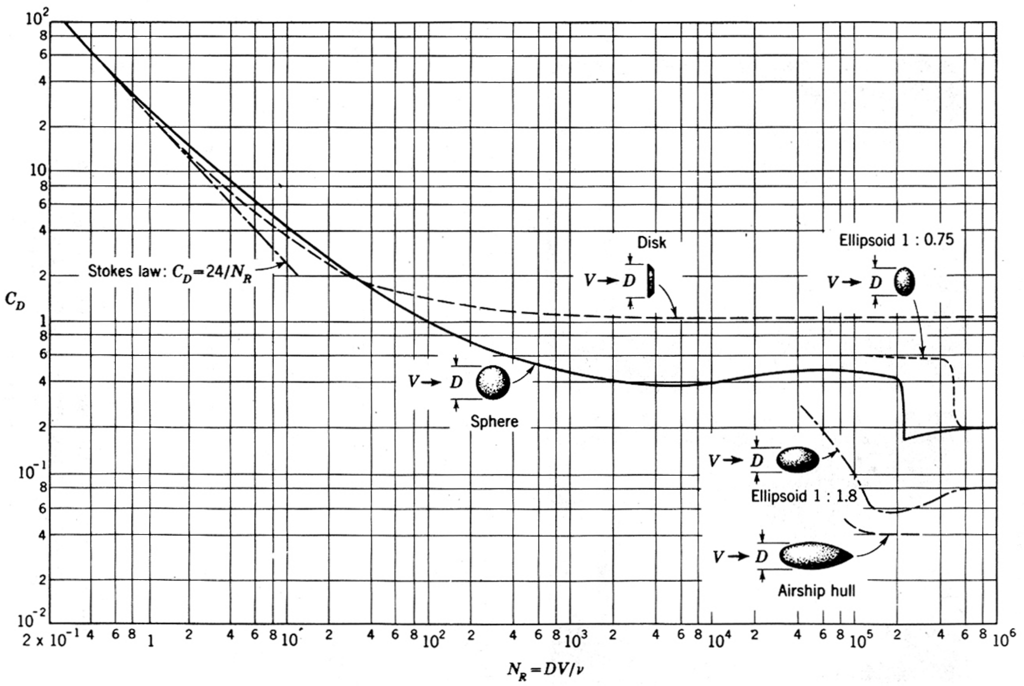
a) a spherical water drop of diameter of 1 mm,

b) a spherical hail of diameter 1 cm,

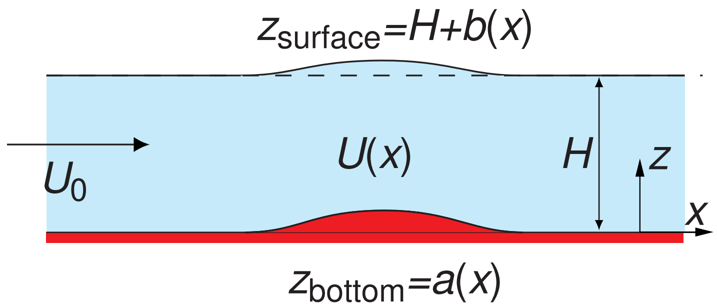
c) a paratrooper with a parachute diameter 11 m and total mass 160 kg,

d) what would be the velocity of the hail (diameter 1 cm) if the Stokes law were valid?

The drag force has the form $F = \frac{1}{2} C_D \rho A V^2$, and the graph gives the coefficient C_D as a function of the Reynolds number $N_R = DV/\nu$. Here A is the cross-sectional area of the object, V its velocity, ρ is the density of the medium, and ν the kinematic viscosity of the medium. (Hint: Since you do not know the Reynolds number in the beginning, make first a simple guess of C_D, and then correct that once you have an estimate of N_R.



1. **Channel flow**
   Consider water flow in a channel, where the bottom has a smooth hump $z = a(x)$. Using mass conservation and Bernoulli equation (simplest at the surface), calculate the rise $b(x)$ of the free surface $z = H + b(x)$ of the water. Assuming both a and b much smaller than H, solve the coefficient c in the linear relation $b(x) = ca(x)$. Is c always positive?



2. **Complex potential**
   Show that $\phi = A(x^2 - y^2)$ satisfies $\nabla^2\phi = 0$ and that $\psi = 2Axy$ gives the same velocity field. Show that ϕ and ψ in this case are real and imaginary parts of the complex function $A(x + iy)^2$.

3. **Velocity field in sound wave**
   By linearizing the Euler equation and the continuity equation, determine the equation for the velocity field $\boldsymbol{v}'$. Show that this has the plane wave solution

   $$\boldsymbol{v}' = A\mathrm{e}^{\mathrm{i}(kx-\omega t)}\boldsymbol{i}$$

   and find how the frequency ω depends on the wave vector k.

4. **Attenuation of sound**
   Formulate the linearized equations for sound wave including also the dissipative term. Note that you have to use the Navier-Stokes equation for compressible fluid. Form a single equation for $\boldsymbol{v}$. Solve this for a plane wave

   $$\boldsymbol{v} = Ae^{i(kx-\omega t)}\boldsymbol{i}.$$

   Keeping k real, show that ω is complex valued and leads to exponential damping of the amplitude of sound, with damping factor $e^{-\Gamma t}$, $\Gamma = \frac{\omega^2}{2c^2\rho_0}(K + \frac{4}{3}\mu)$ to first order in viscosity.

   (Warning: we have here neglected heat conduction, which leads to additional damping of sound.) Estimate the decay time of sound wave in air of frequency $\omega/2\pi = 1$ kHz.