1

$$p_x(x) \propto e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}}, \; p_y(y) \propto e^{-\frac{(y-\bar{y})^2}{2\sigma_y^2}} \;; \text{ x, y independent, } p_{x+y}$$

What is the probabilitydensity for $z = x + y$?

Informal approach to the problem:
The probability that $x \in [x, x + dx]$ and $y \in [y, y + dy]$ is $p_x(x)dx * p_y(y)dy$. When looking at probility for z=x+y, x can be set arbitrary if at the same time fixing y to $y = z - x$. Thus 'summation' of the above probability over x, $\int p_x(x)dx * p_y(z-x)dy = dz * \int p_x(x)dx * p_y(z-x)$, tells the probability that $z \in [z, z + dz]$, and the probability-distribution p_z is given by the integral part.

$$p_z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_x(x)p_y(y)\,\delta(z-x-y)\,dx\,dy = \int_{-\infty}^{\infty} p_x(x)p_y(z-x)\,dx \propto \int_{-\infty}^{\infty} e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}}\, e^{-\frac{(z-x-\bar{y})^2}{2\sigma_y^2}}\,dx$$

Moulding the exponent:

$$-\frac{(x-\bar{x})^2}{2\sigma_x^2} - \frac{(z-x-\bar{y})^2}{2\sigma_y^2} \overset{h:=x-\bar{x}}{=} -\frac{h^2}{2\sigma_x^2} - \frac{(z-h-\bar{x}-\bar{y})^2}{2\sigma_y^2} \overset{a:=z-\bar{x}-\bar{y}}{=} -\frac{h^2}{2\sigma_x^2} - \frac{(a-h)^2}{2\sigma_y^2}$$

$$= -\frac{h^2\sigma_y^2 + \sigma_x^2(h^2 - 2ha + a^2)}{2\sigma_x^2\sigma_y^2} = -\frac{1}{2\sigma_x^2\sigma_y^2}(h^2(\sigma_x^2+\sigma_y^2) - 2ah\sigma_x^2 + \sigma_x^2a^2)$$

$$= -\frac{1}{2\sigma_x^2\sigma_y^2}(h^2(\sigma_x^2+\sigma_y^2) - 2ah\sigma_x^2 + \left(\frac{a\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_y^2}}\right)^2 - \left(\frac{a\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_y^2}}\right)^2 + \sigma_x^2a^2)$$

$$= -\frac{1}{2\sigma_x^2\sigma_y^2}\left(-\left(\frac{a\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_y^2}}\right)^2 + \sigma_x^2a^2\right) - \frac{1}{2\sigma_x^2\sigma_y^2}(h^2(\sigma_x^2+\sigma_y^2) - 2ah\sigma_x^2 + \left(\frac{a\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_y^2}}\right)^2)$$

$$= -\frac{1}{2\sigma_x^2\sigma_y^2}\left(\frac{-a^2\sigma_x^4}{\sigma_x^2+\sigma_y^2} + \sigma_x^2a^2\right) - \frac{1}{2\sigma_x^2\sigma_y^2}\left(h\sqrt{\sigma_x^2+\sigma_y^2} - \frac{a\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_y^2}}\right)^2$$

$$= -\frac{-a^2\sigma_x^4 + (\sigma_x^2+\sigma_y^2)\sigma_x^2a^2}{2\sigma_x^2\sigma_y^2(\sigma_x^2+\sigma_y^2)} - \frac{\sigma_x^2+\sigma_y^2}{2\sigma_x^2\sigma_y^2}(h - \frac{a\sigma_x^2}{\sigma_x^2+\sigma_y^2})^2 \overset{g:=h-\frac{a\sigma_x^2}{\sigma_x^2+\sigma_y^2}}{=} -\frac{-a^2\sigma_x^4 + a^2\sigma_x^4 + a^2\sigma_x^2\sigma_y^2}{2\sigma_x^2\sigma_y^2(\sigma_x^2+\sigma_y^2)} - \frac{\sigma_x^2+\sigma_y^2}{2\sigma_x^2\sigma_y^2}g^2$$

$$= \frac{-a^2}{2\left(\sqrt{\sigma_x^2+\sigma_y^2}\right)^2} - \frac{\sigma_x^2+\sigma_y^2}{2\sigma_x^2\sigma_y^2}g^2$$

Since the first term doesn't depend on x, and $dg = dh$, $dh = dx$, it follows that

$$p_z(z) \propto e^{\frac{-a^2}{2\left(\sqrt{\sigma_x^2+\sigma_y^2}\right)^2}} \int_{-\infty}^{\infty} e^{-\frac{\sigma_x^2+\sigma_y^2}{2\sigma_x^2\sigma_y^2}g^2}\,dg$$

The value of the integral doesn't depend on z, so

$$p_z(z) \propto e^{\frac{-a^2}{2\left(\sqrt{\sigma_x^2+\sigma_y^2}\right)^2}} = e^{-\frac{(z-(\bar{x}+\bar{y}))^2}{2\left(\sqrt{\sigma_x^2+\sigma_y^2}\right)^2}}$$

This is (unnormalised) gaussian distribution with mean $\bar{x}+\bar{y}$ and variance $\sqrt{\sigma_x^2+\sigma_y^2}$

2

2a

The code from previous exercise can be used for obtaining the measurements. Note the difference in the definition of $|M|$: the one used here is the one used in exer 3 multiplied by volume, $|M|_4 = V * |M|_3$.

Settings used in the simulation:

- Update: Metropolis
- Total number of updatesweeps: 2000000
- $\beta = 1.0050525387$
- Number of measurements: 200000 (measurement done on every tenth sweep)
- Average and error calculations were done with errors.c; for $\langle E + |M|\rangle$ and $\langle E - |M|\rangle$, datacolumns $E+|M|$ and $E-|M|$ were constructed. First 1000 measurements where discarded in all cases.

	Value	Error estimate	τ_{int}
$\langle E\rangle$	1675	4.4	60
$\langle \|M\|\rangle$	2400	24	130
$\langle E + \|M\|\rangle$	4080	19	138
$\langle E - \|M\|\rangle$	-730	28	123

(Note: Energy was measured wrong in exercise 3 example code potts_sim.c (There was E $= \sum \delta(s_i, s_j)$ instead of E $= \sum (1 - \delta(s_i, s_j))$). All expectation values containing energy are naturally different if using that code, but since its only a constant shift, the error values shouldn't be affected.)

In the case of statistically independent variables(lecturenotes page 75) the error of sum and difference can be approximated with $\sqrt{(\delta|M|)^2 + (\delta E)^2}$ (note that $\langle E + |M|\rangle = \langle E\rangle + \langle|M|\rangle$). In this case this estimate gives result ≈ 24.4 (the same for both).

2b

The reweighted values for $|M|$ and $|M|^2$ at β' are obtained with

$$\langle |M|\rangle_{\beta'} = \frac{\left\langle |M| e^{-(\beta'-\beta)E}\right\rangle_\beta}{\left\langle e^{-(\beta'-\beta)E}\right\rangle_\beta}, \quad \langle |M|^2\rangle_{\beta'} = \frac{\left\langle |M|^2 e^{-(\beta'-\beta)E}\right\rangle_\beta}{\left\langle e^{-(\beta'-\beta)E}\right\rangle_\beta}$$

Estimated maximum values and locations(evaluated from the plot):

$Size$	β_{max}	χ_{max}
16	0.973	13
32	0.991	43
64	0.999	155

Like in 2a, here also 1000 first measurements were discarded when doing the reweighting.

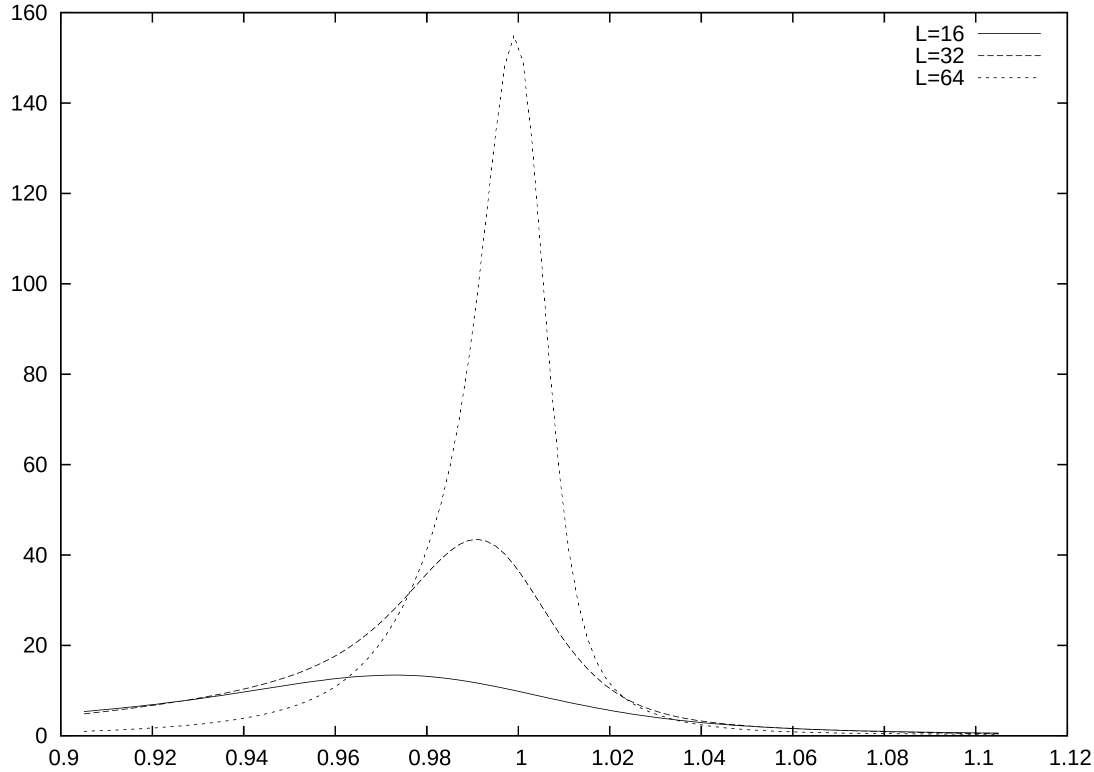

Figure 1: $\chi_M(\beta)$. The behavior is similar to the Ising model case presented in lecturenotes in sec. 9.2. If using the erroneous example code of exercise 3, the maximum shifts left instead of right with increasing lattice size.