Rank in Banach Algebras: A Generalized Cayley-Hamilton Theorem

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Outline of the Talk

1. Some Reminders of Rank, Trace and Determinant
2. The Generalized Characteristic Polynomial
3. The Generalized Cayley-Hamilton Theorem
4. Concluding Remarks
Let $A$ be a complex, semisimple Banach algebra with identity element $1$ and invertible group $A^{-1}$.

For $x \in A$, we denote by $\sigma_A(x) = \{ \lambda \in \mathbb{C} : \lambda 1 - x \notin A^{-1} \}$ and $\sigma'_A(x) = \sigma_A(x) - \{0\}$ the spectrum and nonzero spectrum of $x$, respectively.

If the underlying algebra is clear from the context, then we will agree to omit the subscript $A$ used in the notation above.

This convention will also be followed in the forthcoming definitions of rank, trace, determinant, etc.
Finite-rank elements

Following Aupetit and Mouton, we define the rank of $a \in A$ by

$$\text{rank}_A(a) = \sup_{x \in A} \# \sigma'(xa) \leq \infty.$$ 

In their paper they also showed that the socle, denoted $\text{Soc} A$, of a semisimple Banach algebra $A$ coincides with the collection

$$\mathcal{F} = \{ a \in A : \text{rank}(a) < \infty \}$$

of finite-rank elements.
The Diagonalization Theorem

If \( a \in \text{Soc } A \) has the property that \( \text{rank}(a) = \#\sigma'(a) \), then \( a \) is referred to as a **maximal finite-rank element**.

Maximal finite-rank elements are important because they can be "diagonalized".

Indeed, Aupetit and Mouton’s so-called **Diagonalization Theorem** states that if \( a \in \text{Soc } A \) satisfies \( \text{rank}(a) = \#\sigma'(a) = n \), then \( a \) can be expressed as

\[
a = \lambda_1 p_1 + \ldots + \lambda_n p_n,
\]

where the \( \lambda_i \) are the distinct nonzero spectral values of \( a \), the \( p_i \) are Riesz projections, all of which are minimal (and hence rank one).
Maximal finite-rank elements are dense

In addition, Aupetit and Mouton also showed that the set

\[ E_A(a) = \{ x \in A : \# \sigma'(xa) = \text{rank}(a) \} \]

is dense and open in \( A \).

But this of course means that the set of maximal finite-rank elements is dense in \( \text{Soc} \, A \).
For $a \in \text{Soc } A$, Aupetit and Mouton now use the “spectral” rank to define the \textbf{trace} and \textbf{determinant} as:

\[
\text{Tr}_A(a) = \sum_{\lambda \in \sigma(a)} \lambda m(\lambda, a)
\]

\[
\text{Det}_A(1 + a) = \prod_{\lambda \in \sigma(a)} (1 + \lambda)^{m(\lambda,a)},
\]

where $m(\lambda, a)$ is the \textbf{multiplicity} of $a$ at $\lambda$. 
A brief discription of the notion of **multiplicity** in the abstract case goes as follows:

Let \( a \in \text{Soc } A \), \( \lambda \in \sigma(a) \) and let \( V_\lambda \) be an open disk centered at \( \lambda \) such that \( V_\lambda \) contains no other points of \( \sigma(a) \).

It can be shown that there exists an open ball, say \( U \subseteq A \), centered at \( 1 \) such that \# \( [\sigma(xa) \cap V_\lambda] \) is constant as \( x \) runs through \( E(a) \cap U \).

This constant integer is the multiplicity of \( a \) at \( \lambda \).
To generalize the Cayley-Hamilton Theorem for matrices to the socle of an arbitrary Banach algebra we need a suitable candidate for the characteristic polynomial associated with an element $a \in \text{Soc } A$.

**Definition**

Let $a \in \text{Soc } A$. The **generalized characteristic polynomial** of $a$ is the complex polynomial defined by

$$p_a(\lambda) = \prod_{\alpha \in \sigma_A(a)} (\alpha - \lambda)^{m(\alpha,a)}.$$
The product in the previous definition has a finite number of factors since $a$ belongs to the socle.

Moreover, for any fixed $\lambda \in \mathbb{C}$ it can be shown that $a \mapsto p_a(\lambda)$ is continuous on

$$\mathcal{F}_k = \{a \in \text{Soc } A : \text{rank}(a) \leq k\}$$

for every nonnegative integer $k$. 
The classical case vs. the generalized characteristic polynomial

Example

If \( a \in M_n(\mathbb{C}) \), then it is not necessarily the case that \( p_a(\lambda) \) is equal to the characteristic polynomial as defined in the classical sense. Indeed, if

\[
a = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

then \( p_a(\lambda) = -\lambda (1 - \lambda) \), whereas the classical characteristic polynomial of \( a \) is given by \( q(\lambda) = (-\lambda)^2 (1 - \lambda) \).
The classical case vs. the generalized characteristic polynomial

The previous example shows that the Aupetit-Mouton definition of multiplicity of $0 \in \sigma(a)$ does not necessarily coincide with the algebraic multiplicity associated with the 0 spectral value of a matrix.

However, $p_a(\lambda)$ does in fact coincide with the classical characteristic polynomial when $a$ is an invertible maximal finite-rank matrix.

In order to prove the **generalized Cayley-Hamilton Theorem**, a little preparation is needed:
The subalgebra $pAp$

**Lemma (Aupetit, 2000)**

Let $p$ be a projection of $A$. Then $pAp$ is a closed semisimple subalgebra of $A$ with identity $p$ and

$$\sigma'_{pAp}(pxp) = \sigma'_A(px_p)$$

for each $x \in A$.

**Lemma (Braatvedt-Brits-S., 2015)**

Let $p$ be a finite-rank projection of $A$. Then

$$\text{rank}_{pAp}(pxp) = \text{rank}_A(px_p)$$

for each $x \in A$. 
### Proposition (Braatvedt-Brits-S., 2016)

Let \( A_j = M_{n_j}(\mathbb{C}) \) for each \( j \in \{1, \ldots, k\} \) and let \( A = A_1 \oplus \cdots \oplus A_k \).

Suppose that \( a = (a_1, \ldots, a_k) \) is a maximal finite-rank element of \( A \). Then
\[
\sigma'_{A_i}(a_i) \cap \sigma'_{A_j}(a_j) = \emptyset \quad \text{for } i \neq j.
\]

### Proposition (Braatvedt-Brits-S., 2016)

Let \( A_j = M_{n_j}(\mathbb{C}) \) for each \( j \in \{1, \ldots, k\} \) and let \( A = A_1 \oplus \cdots \oplus A_k \).

Suppose that \( a = (a_1, \ldots, a_k) \) is a maximal finite-rank element of \( A \). Then
\[a_j\] is a maximal finite-rank element of \( A_j \) for each \( j \in \{1, \ldots, k\} \).
 Lemma (Braatvedt-Brits-S., 2016)

Let $A_j = M_{n_j} (\mathbb{C})$ for each $j \in \{1, \ldots, k\}$ and let $A = A_1 \oplus \cdots \oplus A_k$. Suppose that $a = (a_1, \ldots, a_k)$ is an invertible maximal finite-rank element of $A$. Then

$$p_a(\lambda) = \prod_{j=1}^{k} \det A_j (a_j - \lambda 1_j)$$

for all $\lambda \in \mathbb{C}$. 
Let $a \in \text{Soc } A$. For any $x \in A$ we have that

$$p_a(x) = \prod_{\alpha \in \sigma_A(a)} (\alpha 1 - x)^{m(\alpha, a)}.$$
If $a = 0$, the result trivially holds true. So assume that \( \text{rank}(a) = n \geq 1 \).

Moreover, assume first that $a \notin A^{-1}$ is a maximal finite-rank element of $A$.

By the Diagonalization Theorem,

$$ a = \lambda_1 p_1 + \cdots + \lambda_n p_n, $$

and so $a \in B = qAq$, where $q = p_1 + \cdots + p_n$.

Moreover, $a \in B^{-1}$ and $a$ is a maximal finite-rank element of $B$. 

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Sketch of the Proof:

- By the Wedderburn-Artin Theorem, $B$ is isomorphic as an algebra to

$$C = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

- Therefore, if $\phi : B \rightarrow C$ is the aforementioned isomorphism, then

$$p(\lambda) := p_a(\lambda) = -\lambda \prod_{j=1}^{k} \text{Det}_{C_j}(a_j - \lambda 1_j),$$

where $\phi(a) = (a_1, \ldots, a_k)$ and $C_j = M_{n_j}(\mathbb{C})$ for each $j \in \{1, \ldots, k\}$. 
Sketch of the Proof:

Let $\Gamma$ be the union of $n + 1$ disjoint circles with centers respectively at $\lambda_1, \ldots, \lambda_n$ and 0. For each $j \in \{1, \ldots, k\}$ and $\lambda \in \Gamma$ we have

$$(a_j - \lambda 1_j)^{-1} = \frac{1}{\text{Det}_{C_j}(a_j - \lambda 1_j)} b_j(\lambda),$$

where $b_j(\lambda)$ is a $n_j \times n_j$ matrix depending analytically on $\lambda$ since its $(r, m)$-entry is the $(m, r)$-cofactor of $a_j - \lambda 1_j$, and so is a polynomial in $\lambda$. 
Sketch of the Proof:

Now let $e_j(\lambda) \in C$ be the $k$-tuple with $b_j(\lambda)$ in its $j$th component and 0’s elsewhere. Since

$$ (\lambda 1_C - \phi(a))^{-1} = -\left((a_1 - \lambda 1_1)^{-1}, \ldots, (a_k - \lambda 1_k)^{-1}\right), $$

it follows from the Holomorphic Functional Calculus that

$$ p(\phi(a)) = \frac{1}{2\pi i} \sum_{j=1}^{k} \left[ \int_{\Gamma} \lambda \left( \prod_{i \neq j} \text{Det} C_i (a_i - \lambda 1_i) \right) e_j(\lambda) \, d\lambda \right]. $$
Sketch of the Proof:

- Using the standard basis for $C_j$ and Cauchy’s Theorem, it follows that
  \[ \int_{\Gamma} \lambda \left( \prod_{i \neq j} \text{Det}_{C_i} (a_i - \lambda 1_i) \right) e_j(\lambda) \, d\lambda = 0 \]
  for each $j \in \{1, \ldots, k\}$.

- Thus, $p(\phi(a)) = 0$, and so, $p(a) = 0$ in $B$ and hence in $A$. This is $p(\lambda)$ does not have a constant term; so the fact that $q \neq 1$ does not matter here.

- Of course, if $a$ above belonged to $A^{-1}$, then $\text{Soc} \ A = A$ so that $A$ itself is finite-dimensional. Consequently, we may apply our argument with the Wedderburn-Artin Theorem to $A$ directly and reach the same conclusion.
Sketch of the Proof:

- We have therefore shown that $p_a(a) = 0$ if $a$ is a maximal finite-rank element.
- Suppose now that $a$ is not a maximal finite-rank element.
- By the density of $E(a)$, let $(x_m) \subseteq E(a)$ such that $x_m \to 1$ as $m \to \infty$, and let $p_m(\lambda)$ be the characteristic polynomial of $x_m a$.
- Then $p_m(x_m a) = 0$ for all $m$.
- Finally, using the Lebesgue Dominated Convergence Theorem, it can be shown that $p_m(x_m a) \to p(a)$ as $m \to \infty$, which completes the proof.
Concluding remarks

In view of the Aupetit and Mouton definition of the determinant it is very tempting to define

\[
\text{Det}_A (a - \lambda \mathbf{1}) := \prod_{\alpha \in \sigma_A(a)} (\alpha - \lambda)^{m(\alpha, a)}.
\]

By definition one would then have, as in the case for matrices,

\[
p_a(\lambda) = \text{Det}_A (a - \lambda \mathbf{1}).
\]

The main reasons that this route was not taken are the following:
Axler’s (somewhat controversial) paper entitled “Down with determinants!”.

To avoid possible confusion – the example we studied earlier clearly makes the point.

Since this definition does not have all the properties of the classical determinant. Indeed, this determinant is not multiplicative. (It is easy to construct an example of this in $\mathbb{C}^3$).
Concluding remarks

It should be mentioned however that this problem does not surface in the Aupetit-Mouton formulation, that is

$$\det_A [(1 + a)(1 + b)] = \det_A (1 + a) \det_A (1 + b)$$

for all $a, b \in \text{Soc } A$.

Moreover, a generalized Sylvester’s Theorem also holds true:

**Theorem (Braatvedt-Brits-S., 2015)**

*Let $a \in \text{Soc } A$ and let $b \in A$. Then*

$$\det_A (1 + ab) = \det_A (1 + ba).$$


