Approximate Identities, Factorization, and Amenability in Algebras of Random Elements

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Outline of the Talk

1. The Algebra of Random Elements
2. Approximate Identities
3. Factorization
4. Amenability
Probability space \((\Omega, \mathcal{F}, \mu)\) complete, no atoms, Banach space \(X\)

\[ L_0(\Omega; X) = L_0(\Omega, \mathcal{F}, \mu; X) = \text{all } X\text{-valued Bochner-measurable functions on } \Omega, \text{ topology of convergence in probability.} \]

\[ L_0(\Omega) = L_0(\Omega; \mathbb{C}) \]

Consider \(L_0(\Omega; X)\) as a module over \(L_0(\Omega)\).

Write \(x \in L_0(\Omega; X)\), \(x \in X\), identify \(x\) with the constant element in \(L_0(\Omega; X)\).

Convergence in probability is metrizable. A complete metric:

\[ d_0(x, y) = \mathbb{E}(\min\{\|x - y\|, 1\}) \]
Recall that a subset $E$ of $L_0(\Omega; X)$ is bounded in the topological vector space sense if and only if for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\mu[\|x\| \geq M_\varepsilon] < \varepsilon, \quad x \in E.$$ 

Such sets are usually called \textit{stochastically bounded}.

A map $T : L_0(\Omega; X) \to L_0(\Omega; Y)$ is called \textit{modular} if

$$T(\varphi x) = \varphi T x, \quad \varphi \in L_0, \ x \in L_0(\Omega; X).$$

\textbf{Proposition}

\textit{If $T : L_0(\Omega; X) \to L_0(\Omega; Y)$ is continuous and modular, then for all $x \in L_0(\Omega; X)$,

$$ (Tx)(\omega) = T(x(\omega))(\omega) \ a.s. \quad (1) $$}
Proof.

If

\[ x = \sum_{i=1}^{n} x_i \chi_{E_i} \]

is a simple element of \( L_0(\Omega; X) \), then

\[ Tx = \sum_{i=1}^{n} (Tx_i) \chi_{E_i} , \]

so (1) holds for such \( x \). Since every element of \( L_0(\Omega; X) \) is a limit a.s. of simple elements and convergence in probability implies convergence a.s. of a subsequence, (1) follows for all \( x \) by continuity. \( \square \)
A a Banach algebra with $e$. $L_0(\Omega; A)$ is a Fréchet algebra which is not locally convex (F-algebra). We study its properties as a topological algebra and its relationship to $A$.

**Motivation:**
Early work with Leon Brown on stochastic continuity algebras

**Theorem (Random Johnson-Sinclair Theorem, Velasco/Villena, 1995))**

*Let $A$ be semisimple. Then every derivation from $A$ to $L_0(\Omega; A)$ is continuous.*

In earlier work with M.V. Velasco, we studied such topics as the appropriate notion of spectrum in $L_0(\Omega; A)$, the radical, ideal theory, and automatic continuity of homomorphisms. Here we go into some other topics.
For $X$ a Banach left $A$-module, assume $\|a \cdot x\| \leq \|a\| \|x\|$, $a \in A$, $x \in X$. One can check easily that $L_0(\Omega; X)$ becomes a left topological $L_0(\Omega; A)$- and $L_0(\Omega)$-module under the natural pointwise almost everywhere operations, meaning that the the map $(a, x) \mapsto a \cdot x$ is jointly continuous. Similarly for right modules and bimodules $X$. 
Assume $A$ is nonunital, fixed for the remainder of this and the next section.

We shall call a subset $E$ of $L_0(\Omega; A)$ \textit{dominated} if for some $m \in L_0(\Omega)$, $\|a\| \leq m$ a.s. for all $a \in E$.

\textbf{Theorem}

\textit{The following are equivalent:}

(i) $L_0(\Omega; A)$ has a (stochastically) bounded left [right, two-sided] approximate identity.

(ii) $L_0(\Omega; A)$ has a dominated left [right, two-sided] approximate identity.

(iii) $A$ has a bounded left [right, two-sided] approximate identity.
Proof.

It suffices to deal with left approximate identities. If \( \{u_\alpha\} \) is a bounded left approximate identity for \( A \), then the corresponding set of constants is a left approximate identity for \( L_0(\Omega; A) \). Indeed, suppose that \( \|u_\alpha\| \leq M \) for all \( \alpha \) and some \( M \geq 1 \). Given \( a \in L_0(\Omega; A) \) and \( 0 < \varepsilon < 1 \), choose a simple element \( b \in L_0(\Omega; A) \) with \( \mu[\|a - b\| \geq \varepsilon/M] < \varepsilon \). Clearly there exists \( \alpha_0 \) such that \( \|u_\alpha b - b\| < \varepsilon \) a.s. for all \( \alpha \geq \alpha_0 \). And \( \|u_\alpha(a - b)\| < \varepsilon \) wherever \( \|a - b\| < \varepsilon/M \). Hence for all \( \alpha \geq \alpha_0 \), \( \|u_\alpha a - a\| < 3\varepsilon \) outside of a set of measure at most \( 2\varepsilon \). Of course, the set \( \{u_\alpha\} \) is dominated, so (iii) implies (ii), which clearly implies (i).

Suppose that \( \{u_\alpha\} \) is a bounded left approximate identity in \( L_0(\Omega; A) \). Then there exists \( M > 0 \) such that \( \mu[\|u_\alpha\| \geq M] < 1/4 \) for all \( \alpha \). For each \( \alpha \) choose \( E_\alpha \in \mathcal{F} \) such that \( \|u_\alpha\| < M \) a.s. on \( E_\alpha \) and \( \mu(E_\alpha) = 1/2 \).
Define $u_\alpha$ by the vector integral

$$
u_\alpha = 2 \int_{E_\alpha} u_\alpha \, d\mu,$$

so $\|u_\alpha\| \leq M$. Note that for $a \in A$ and $b \in L_0(\Omega; A)$ with $E\|b\| < \infty$,

$$\left(\int_{\Omega} b \, d\mu\right) a = \int_{\Omega} ba \, d\mu.$$

This is clear when $b$ is a simple function, and one can pass to limits of uniformly bounded sequences in the Bochner integral. Thus if $\varepsilon > 0$ and $a \in A$, choose $\alpha_0$ so that $\mu(A_{\alpha,\varepsilon}) < \varepsilon$ for all $\alpha \geq \alpha_0$, where

$$A_{\alpha,\varepsilon} = \{\omega: \|u_\alpha(\omega)a - a\| \geq \varepsilon\}.$$
If $\alpha \geq \alpha_0$, then

$$
\|u_\alpha a - a\| = \left\| 2 \int_{E_\alpha} u_\alpha a \ d\mu - 2 \int_{E_\alpha} a \ d\mu \right\| \\
\leq 2 \int_{E_\alpha} \|u_\alpha a - a\| \ d\mu \\
= 2 \int_{E_\alpha \cap A_{\alpha, \varepsilon}} \|u_\alpha a - a\| \ d\mu + 2 \int_{E_\alpha \setminus A_{\alpha, \varepsilon}} \|u_\alpha a - a\| \ d\mu \\
< 2\left[ (M + 1)\|a\| + 1 \right]\varepsilon.
$$

So (i) implies (iii).
X a left Banach $A$-module, so $L_0(\Omega; X)$ is a topological left module over $L_0(\Omega; A)$.

Let $\Sigma(A, X) = \text{closure in } L_0(\Omega; X)$ of all sums of the form $\sum_{i=1}^{n} a_i \cdot x_i$.

The celebrated factorization theorem of Cohen, Hewitt, Allan, and Sinclair has a version in the present context.
Theorem (Stochastic Factorization Theorem)

Suppose that $A$ has no identity, but $L_0(\Omega; A)$ has a stochastically bounded left approximate identity, and that $X$ is a left Banach $A$-module. Let $x \in \Sigma(A, X)$, and let $X_0$ be a closed, separable subset of $X$ such that $x \in X_0$ a.s. and the values of $x$ outside of some null set are dense in $X_0$. Then there is a separable, closed subalgebra $A_0$ of $A$ with a bounded sequential left approximate identity $\{u_n\}$ such that $x \in \Sigma(A_0, X_0)$ a.s. and $u_ny \to y$, $y \in X_0$. 

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Let $1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \to \infty$. Then for some $M > 0$ and any $N \geq 1$ and $\varepsilon > 0$, there exist $a \in L_0(\Omega; A)$ with $\|a\| \leq M$ a.s. and $y_n \in L_0(\Omega; A) \cdot x$, $n = 1, 2, \ldots$ such that:

(i) $y_n \in \Sigma(A_0, X_0)$ a.s., $n \geq 1$;

(ii) $x = a^n \cdot y_n$ a.s., $n \geq 1$;

(iii) $\|x - y_n\| \leq \varepsilon$ a.s., $n = 1, \ldots, N$;

(iv) $\|y_n\| \leq \alpha_n^n \|x\|$ a.s., $n \geq 1$. 

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The proof of this theorem rests on its well-known version for Banach algebras and an appropriately applied selection theorem.

**Lemma**

Suppose that $A$ has a bounded left approximate identity $\{u_\alpha\}$ and $x \in \Sigma(A, X)$. Then $u_\alpha \cdot x \to x$ in $L_0(\Omega; X)$. 

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Proof.

Assume $\|u_\alpha\| \leq M$ for all $\alpha$. Given $\varepsilon > 0$, choose

$$a_1, \ldots, a_n \in L_0(\Omega; A), \quad y_1, \ldots, y_n \in L_0(\Omega; X)$$

such that

$$\mu \left[ \|x - \sum_{i=1}^{n} a_i \cdot y_i\| \geq \varepsilon \right] < \varepsilon.$$

Set $y = \sum_{i=1}^{n} a_i \cdot y_i$ and $\delta = \varepsilon/n$. Since the indices $\alpha$ are directed, for all sufficiently large $\alpha$ and each $i$, outside of a set $E_i$ of measure at most $\delta$, we can get

$$\mu[\|u_\alpha a_i \cdot y_i - a_i \cdot y_i\| \geq \delta] < \delta.$$

Then for such $\alpha$ and outside of the set $E_1 \cup \ldots \cup E_n$ of measure at most $\varepsilon$,

$$\mu[\|u_\alpha \cdot y - y\| \geq \varepsilon] \leq \sum_{i=1}^{n} \mu[\|u_\alpha a_i \cdot y_i - a_i \cdot y_i\| \geq \delta] < \varepsilon,$$

A standard $3\varepsilon$ argument finishes the proof. $\square$
Lemma

Let \( x \in \Sigma(A, X) \), and let \( X_0 \) be a closed, separable subset of \( X \) such that \( x \in X_0 \) a.s. and the values of \( x \) outside of some null set are dense in \( X_0 \). If \( A \) has a left approximate identity bounded by \( M \), then there exists a separable, closed subalgebra \( A_0 \) of \( A \) with a sequential left approximate identity \( \{u_n\} \) bounded by \( M \) such that \( x \in \Sigma(A_0, X_0) \) a.s. and \( u_n y \to y, \ y \in X_0 \).

Proof.

Let \( \{u_\alpha\} \) be a bounded left approximate identity for \( A \), so \( u_\alpha \cdot x \to x \) by the lemma above. Since \( L_0(\Omega; X) \) is metrizable, there is a countable subnet \( \{u_{\alpha_m}\}_{m=1}^\infty \) of \( \{u_\alpha\} \) such that \( u_{\alpha_m} \cdot x \to x \) in \( L_0(\Omega; X) \), and by passing to a subsequence we may assume that \( u_{\alpha_m} \cdot x \to x \) a.s. By considering the complement of some null set in \( \Omega \), assume that the values of \( x \) are dense in \( X_0 \) and \( u_{\alpha_m} \cdot x \to x \) everywhere.
If $x_1, x_2, \ldots$ is a dense subset of $X_0$ consisting of values of $x$, then 
$u_{\alpha_m}x_j \to x_j$ for all $j$. Hence $u_n y \to y$, $y \in X_0$. Finally, apply a 
well-known argument to conclude that there is a closed, separable 
subalgebra $A_0$ of $A$ with a sequential left approximate identity containing 
$\{u_n : n = 1, 2, \ldots\}$. □

**Proof of the Theorem.**

We have $X_0 \subset \Sigma(A_0, X_0)$ and $\Sigma(A_0, X_0)$ is separable. Let

$$A_M = \{a \in A_0 : \|a\| \leq M\},$$

and let $\Pi$ be the complete, separable metric space

$$\Pi = A_M \times \prod_{n=1}^{\infty} \Sigma(A_0, X_0)_n.$$
For $x \in X_0$, set

$$\Phi(x) = \{(a, y_1, y_2, \ldots) \in \Pi : x = a^n \cdot y_n; \|y_n\| \leq \alpha_n^n \|x\| \ \forall \ n \geq 1; \|x - y_n\| \leq \varepsilon, 1 \leq n \leq N\}$$

For each $x \in X_0$, $\Phi(x)$ is clearly closed in $\Pi$, and the usual factorization theorem asserts that it is nonempty.

Let $K$ be compact in $\Pi$, and as usual in the theory of multifunctions, let

$$\Phi^{-1}(K) = \{x : \Phi(x) \cap K \neq \emptyset\}.$$ 

One can check that $\Phi^{-1}(K)$ is closed in $X_0$ for all compact $K$ in $\Pi$, so we may apply a known selection theorem to conclude that there is a measurable selection $\varphi$ for $\Phi$, i.e., a measurable map $\varphi : X_0 \to \Pi$ such that $\varphi(x) \in \Phi(x), \ x \in X_0$. Our theorem now follows by taking, as random variables,

$$(a, y_1, y_2, \ldots) = \varphi(x).$$
Amenability

An algebra, $X$ an $A$-bimodule. A *derivation* on $A$ to $X$ is a linear map $D : A \to X$ satisfying

$$D(ab) = D(a) \cdot b + a \cdot D(b).$$

We are interested in the case where $A$ is a unital Banach algebra, $X$ is a Banach $A$-bimodule, and

$$D : L_0(\Omega; A) \to L_0(\Omega; X)$$

is a continuous derivation.

First observation:

**Proposition**

*Every continuous derivation* $D : L_0(\Omega; A) \to L_0(\Omega; X)$ *is modular.*
Proof.

If \( \chi_E \in L_0(\Omega) \) is a characteristic function, then \( \chi_E^2 = \chi_E \), so

\[
D(\chi_E) = D(\chi_E^2) = D(\chi_E)\chi_E + \chi_E D(\chi_E) = 2\chi_E D(\chi_E).
\]

I.e., \( (1 - 2\chi_E) D(\chi_E) = 0 \) a.s. Since \( (1 - 2\chi_E) \neq 0 \) on \( \Omega \), we have \( D(\chi_E) = 0 \). So if \( a \in A \), then \( D(a\chi_E) = D(a)\chi_E \). Now by linearity, we have

\[
D(\varphi a) = \varphi D(a)
\]

for any simple function \( \varphi \in L_0(\Omega) \) and simple element \( a \in L_0(\Omega; A) \). Recall that a.s. convergence implies convergence in probability, and convergence in probability implies a.s. convergence of a subsequence. Thus using the continuity of \( D \), recalling that elements of \( L_0(\Omega; A) \) are Bochner measurable, and passing to limits first in \( \varphi \) and then in \( a \), we obtain (2) or all \( \varphi \) and \( a \).
Definition

1. A stochastic derivation on $L_0(\Omega; A)$ is a continuous derivation $D : L_0(\Omega; A) \to L_0(\Omega; X)$ for some Banach $A$-bimodule $X$.

2. Let $Z^1(A, X)$ denote the space of all continuous derivations from $A$ to $X$. By a random derivation on $A$ we mean an element of $L_0(\Omega; Z^1(A, X))$. By abuse of language, we call a stochastic derivation $D$ random if there exists $D \in L_0(\Omega; Z^1(A, X))$ such that

\[ D(a) = D(a) \text{ a.s., } a \in L_0(\Omega; A) \]

(i.e. $D(a)(\omega) = D(\omega)(a(\omega))$).

3. A stochastic derivation is inner if for some $x \in L_0(\Omega; X)$,

\[ D(a) = x \cdot a - a \cdot x. \]
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Definition

- $L_0(\Omega; A)$ is \textit{weakly amenable} if for every $A$-bimodule $X$ [$X = A$], every stochastic derivation on $L_0(\Omega; A)$ into $L_0(\Omega; X^*)$ is inner.

- $L_0(\Omega; A)$ is \textit{randomly weakly amenable} if for every $A$-bimodule $X$ [$X = A$], every random derivation on $L_0(\Omega; A)$ into $L_0(\Omega; X^*)$ is inner.

Theorem

Consider the following conditions:

- $L_0(\Omega; A)$ is \textit{weakly amenable}.
- $L_0(\Omega; A)$ is randomly \textit{weakly amenable}.
- $A$ is \textit{weakly amenable}.

Then (1) $\implies$ (2) $\implies$ (3). $\implies$ (2).
Proof.

Clearly (1) \implies (2).

(2) \implies (3). Let $D$ be a nonzero continuous derivation on $X^*$. Then $D$ extends to a constant element of $L_0(\Omega; Z^1(A, X))$; denote the derived derivation on $L_0(\Omega; A)$ by $D$. By hypothesis, this derivation is inner, Choose $x^* \in L_0(\Omega; X^*)$ such that

$$D(a) = x^* \cdot a - a \cdot x^*, \quad a \in L_0(\Omega; A).$$

Choose a set $E \subset \Omega$ such that $\int_E x^* \, d\mu \neq 0$, and let

$$x^* = \frac{1}{\mu(E)} \int_E x^* \, d\mu.$$

Considering $a \in A$ as a constant element of $L_0(\Omega; A)$ and hence $D(a)$ as a constant element of $L_0(\Omega; X^*)$, we have using the Bochner integral,

$$D(a) = \frac{1}{\mu(E)} \int_E D(a) \, d\mu = \frac{1}{\mu(E)} \int_E (x^* \cdot a - a \cdot x^*) \, d\mu$$

$$= \left( \frac{1}{\mu(E)} \int_E x^* \, d\mu \right) \cdot a - a \cdot \left( \frac{1}{\mu(E)} \int_E x^* \, d\mu \right) = x^* \cdot a - a \cdot x^*, \quad a \in A.$$
(3) \implies (2). Since every continuous derivation on $L_0(\Omega; A)$ is determined by its values at constants, it suffices to deal with derivations from $A$ to $L_0(\Omega; X^*)$.

For $X^*$ a Banach dual $A$-module and $x^* \in X^*$, let $\delta_{x^*}$ be the inner automorphism on $A$ determined by $x^*$, and let

$$
C(A, X^*) = \{ x^* \in X^* : x^* \cdot a = a \cdot x^* \ \forall a \in A \} = \{ x^* : \delta_{x^*} = 0 \}.
$$

Then $\delta_{x^*} = \delta_{y^*}$ if and only if $x^* - y^* \in C(A, X^*)$. And for all $x^*$, $\| \delta_{x^*} \| \leq 2 \| x^* \|$.

So there is a linear bijection $\Phi : Z^1(A, X^*) \to X^*/C(A, X^*)$, since $A$ is amenable, which is easily seen to be a topological isomorphism. Moreover, if $D \in L_0(\Omega; Z^1(A, X^*))$, there is a separable subspace $S$ of $Z^1(A, X^*)$ such that $D \in S$ a.s. So the space $\Phi(S)$ is separable. After checking one technical point, it follows from a classical selection theorem that there is an $x^* \in L_0(\Omega; X^*)$ such that $D = \delta_{x^*}$ a.s.
Kiitos — Thank you for your attention!