Arens-Michael envelopes of Ore extensions

Peter Kosenko

Higher School of Economics

July 11, 2017
Arens-Michael algebras

All algebras are assumed to be complex, unital and associative.

**Definition**

An Arens-Michael algebra is a complete locally convex algebra with topology that can be generated by a family of submultiplicative seminorms.

Here are some examples:

1. Any Banach algebra is an Arens-Michael algebra.
2. For a locally compact topological space $X$ the algebra of continuous functions $C(X)$, endowed with the compact-open topology, is an Arens-Michael algebra.
3. For any open $U \subset \mathbb{C}^n$ the algebra of holomorphic functions $\mathcal{O}(U)$, endowed with the compact-open topology, is an Arens-Michael algebra.
Arens-Michael envelopes

Definition

Let $A$ be an algebra. The Arens-Michael envelope of $A$ is a pair $(\hat{A}, i_A)$, where $\hat{A}$ is an Arens-Michael algebra and $i_A : A \to \hat{A}$ is an algebra homomorphism, satisfying the following universal property: for any Arens-Michael algebra $B$ and algebra homomorphism $\varphi : A \to B$ there exists a unique continuous algebra homomorphism $\hat{\varphi} : \hat{A} \to B$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\hat{A} & \xrightarrow{\hat{\varphi}} & B \\
\uparrow i_A & & \downarrow \varphi \\
A & & 
\end{array}
$$

The Arens-Michael envelope of any algebra exists and unique. It is isomorphic to the completion of $A$ with respect to the family of all submultiplicative seminorms on $A$. Also notice that due to the Arens-Michael decomposition it suffices to check the universal property when $B$ is a Banach algebra.
Examples

**Theorem (J. Taylor, 1972)**

The Arens-Michael envelope of $\mathbb{C}[t_1, \ldots t_n]$ is isomorphic to $O(\mathbb{C}^n)$. 

**Proposition (J. Taylor, 1972)**

Let $F_n$ be the free algebra on $n$ non-commuting variables $x_i$. The Arens-Michael envelope of $F_n$ can be described as follows:

$$\hat{F}_n \cong \left\{ f = \sum_{w \in W_n} a_w x_w : ||f||_{\rho} = \sum_{w \in W_n} |a_w|^\rho |w| < \infty, 0 \leq \rho < \infty \right\},$$

where $W_n = \bigotimes_{d=0}^{\infty} \{1, \ldots, n\}$. The resulting Arens-Michael algebra will be denoted by $F_n$.

**Theorem (J. Taylor, 1972)**

Let $g$ be a finite-dimensional semisimple Lie algebra. The Arens-Michael envelope of $U(\mathfrak{g})$ is isomorphic to the direct product $\prod_{\hat{\mathfrak{g}}} \text{Mat}(V)$, where $\hat{\mathfrak{g}}$ is the set of finite dimensional irreducible reps of $g$ (or the equivalence classes, to be exact).
Examples

Theorem (J. Taylor, 1972)

The Arens-Michael envelope of $\mathbb{C}[t_1, \ldots, t_n]$ is isomorphic to $O(\mathbb{C}^n)$.

Proposition (J. Taylor, 1972)

Let $F_n$ be the free algebra on $n$ non-commuting variables $x_i$. The Arens-Michael envelope of $F_n$ can be described as follows:

$$\hat{F}_n \simeq \{ f = \sum_{w \in W_n} a_w x^w : \| f \|_\rho = \sum_{w \in W_n} |a_w| \rho^{|w|} < \infty, 0 < \rho < \infty \},$$

where $W_n = \bigsqcup_{d=0}^{\infty} \{1, \ldots, n\}^d$ and $x^w = x_{w_1} x_{w_2} \ldots x_{w_n}$. The resulting Arens-Michael algebra will be denoted by $\mathcal{F}_n$. 
Examples

**Theorem (J. Taylor, 1972)**

*The Arens-Michael envelope of $\mathbb{C}[t_1, \ldots t_n]$ is isomorphic to $O(\mathbb{C}^n)$.***

**Proposition (J. Taylor, 1972)**

Let $F_n$ be the free algebra on $n$ non-commuting variables $x_i$. The Arens-Michael envelope of $F_n$ can be described as follows:

$$\widehat{F}_n \simeq \{ f = \sum_{w \in W_n} a_w x^w : \|f\|_\rho = \sum_{w \in W_n} |a_w| \rho^{|w|} < \infty, 0 < \rho < \infty \},$$

where $W_n = \bigsqcup_{d=0}^{\infty} \{1, \ldots, n\}^d$ and $x^w = x_{w_1} x_{w_2} \ldots x_{w_n}$. The resulting Arens-Michael algebra will be denoted by $\tilde{F}_n$.

**Theorem (J. Taylor, 1972)**

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra. The Arens-Michael envelope of $U(\mathfrak{g})$ is isomorphic to the direct product $\prod_{V \in \hat{\mathfrak{g}}} \text{Mat}(V)$, where $\hat{\mathfrak{g}}$ is the set of finite dimensional irreducible reps of $\mathfrak{g}$ (or the equivalence classes, to be exact).
The Arens-Michael envelope is a left adjoint functor to the forgetful functor, therefore it commutes with quotients. In other words, we have the following theorem:

**Theorem**

Let $A$ be an algebra and suppose that $I \subset A$ is a two-sided ideal. Denote by $J$ the closure of $i_A(I)$ in $\hat{A}$. Then $J$ is a two-sided ideal in $\hat{A}$ and the induced homomorphism $A/I \to \hat{A}/J$ extends to a topological algebra isomorphism

$$(\hat{A}/I) \cong (\hat{A}/J)$$
The Arens-Michael envelope is a left adjoint functor to the forgetful functor, therefore it commutes with quotients. In other words, we have the following theorem:

**Theorem**

Let $A$ be an algebra and suppose that $I \subseteq A$ is a two-sided ideal. Denote by $J$ the closure of $i_A(I)$ in $\hat{A}$. Then $J$ is a two-sided ideal in $\hat{A}$ and the induced homomorphism $A/I \rightarrow \hat{A}/J$ extends to a topological algebra isomorphism

$$\hat{(A/I)} \cong (\hat{A}/\hat{J})$$

As a corollary (not quite trivial), we get the following result:

**Theorem**

For any complex affine algebraic variety $X$ we have

$$\hat{(O_{reg}(X))} \cong O_{hol}(X).$$
It turns out that $i_A : A \rightarrow \hat{A}$ isn’t always injective!
It turns out that $i_A : A \to \hat{A}$ isn’t always injective!

**Proposition**

Let $A = \mathbb{C} \langle x, y \rangle / ([x, y] - 1)$. Then $\hat{A} \simeq 0$.

The proof relies on the fact that if $x, y \in B$, where $B$ is a non-zero Banach algebra, then $[x, y] \neq 1$. 
It turns out that $i_A : A \to \hat{A}$ isn’t always injective!

**Proposition**

Let $A = \mathbb{C} \langle x, y \rangle / ([x, y] - 1)$. Then $\hat{A} \simeq 0$.

The proof relies on the fact that if $x, y \in B$, where $B$ is a non-zero Banach algebra, then $[x, y] \neq 1$.

As a matter of fact, let us formulate the following proposition:

**Proposition**

Consider an element $f \in F_n$. Then the following statements are equivalent:

1. there exist a Banach algebra $B$ and elements $x_1, \ldots, x_n \in B$ such that $f(x) = 0$.
2. if $I \subset F_n$ is the smallest two-sided ideal which contains $f$, then $(F_n/I) \neq 0$.

Such elements are called *admissible*. For example, $xy - yx - 1$ is not admissible.
It turns out that $i_A : A \to \hat{A}$ isn’t always injective!

Proposition

Let $A = \mathbb{C} \langle x, y \rangle / ([x, y] - 1)$. Then $\hat{A} \simeq 0$.

The proof relies on the fact that if $x, y \in B$, where $B$ is a non-zero Banach algebra, then $[x, y] \neq 1$.

As a matter of fact, let us formulate the following proposition:

Proposition

Consider an element $f \in F_n$. Then the following statements are equivalent:

1. there exist a Banach algebra $B$ and elements $x_1, \ldots, x_n \in B$ such that $f(x) = 0$.
2. if $I \subset F_n$ is the smallest two-sided ideal which contains $f$, then
   \[
   (\widehat{F_n/I}) \neq 0.
   \]

Such elements are called *admissible*. For example, $xy - yx - 1$ is not admissible.
Ore extensions

Definition

Let $A$ be an algebra. For any endomorphism $\alpha : A \to A$ denote by $A_\alpha$ an $A$-bimodule, which coincides with $A$ as a left $A$-module, and whose structure of right $A$-module is defined by $x \circ a = x\alpha(a)$, where $x \in A_\alpha$, $a \in A$.

In a similar fashion one defines $\alpha A$. 
Ore extensions

**Definition**

Let $A$ be an algebra. For any endomorphism $\alpha : A \to A$ denote by $A_\alpha$ a an $A$-bimodule, which coincides with $A$ as a left $A$-module, and whose structure of right $A$-module is defined by $x \circ a = x\alpha(a)$, where $x \in A_\alpha$, $a \in A$. In a similar fashion one defines $\alpha A$.

Let us recall the definition of Ore extensions:

**Definition**

Let $A$ be an algebra, $\alpha \in \text{End}(A)$ and $\delta \in \text{Der}(A, \alpha A)$, in other words,

\[ \delta(ab) = \alpha(a)\delta(b) + \delta(a)b. \]

Then the Ore extension of $A$ wrt $\alpha$ and $\delta$ is the vector space $A[t; \alpha, \delta] = \{ \sum_{i=0}^{n} a_i t^i : a_i \in A \}$ with the multiplication, which is uniquely determined by

\[ ta = \alpha(a)t + \delta(a) \quad (a \in A). \]
Ore extensions

Definition

Let $A$ be an algebra. For any endomorphism $\alpha : A \to A$ denote by $A_{\alpha}$ an $A$-bimodule, which coincides with $A$ as a left $A$-module, and whose structure of right $A$-module is defined by $x \circ a = x\alpha(a)$, where $x \in A_{\alpha}, a \in A$. In a similar fashion one defines $\alpha A$.

Let us recall the definition of Ore extensions:

Definition

Let $A$ be an algebra, $\alpha \in \text{End}(A)$ and $\delta \in \text{Der}(A, \alpha A)$, in other words,

$$\delta(ab) = \alpha(a)\delta(b) + \delta(a)b.$$ 

Then the Ore extension of $A$ wrt $\alpha$ and $\delta$ is the vector space $A[t; \alpha, \delta] = \{\sum_{i=0}^{n} a_i t^i : a_i \in A\}$ with the multiplication, which is uniquely determined by

$$ta = \alpha(a)t + \delta(a) \quad (a \in A).$$

Also, if $\delta = 0$ and $\alpha$ is invertible, then one can define the Laurent Ore extension $A[t, t^{-1}; \alpha] = \{\sum_{i=-n}^{n} a_i t^i : a_i \in A\}$ with the multiplication defined similarly.
Definition
Let $A$ is an Arens-Michael algebra. A family of continuous linear maps $\mathcal{T} \subset \mathcal{L}(E)$ is called $m$-localizable $\iff$ there exists a generating family of submultiplicative seminorms $\{|| \cdot ||_\nu : \nu \in \Lambda\}$ on $A$ such that for every $T \in \mathcal{T}$ and $\nu \in \Lambda$ there exists $C > 0$ such that $||T(x)||_\nu \leq C||x||_\nu$ for every $x \in A$. Theorem (A. Pirkovskii)
Let $A$ be an algebra, $\alpha \in \text{End}(A)$ and $\delta \in \text{Der}(A, \alpha A)$, such that $(\hat{\alpha}, \hat{\delta})$ is an $m$-localizable pair of morphisms $\hat{A} \to \hat{A}$. Fix a generating family of seminorms $\{|| \cdot ||_\lambda : \lambda \in \Lambda\}$ on $\hat{A}$. Then the locally convex space $O(C, \hat{A}) = \{f = \sum_{k=0}^{\infty} a_k z^k : \sum_{k=0}^{\infty} ||a_k||_\lambda \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty\}$ admits a unique multiplication, which is induced from $A[z; \alpha, \delta]$, such that the space $O(C, \hat{A})$ becomes an Arens-Michael algebra, which is denoted by $O(C, \hat{A}; \hat{\alpha}, \hat{\delta})$, and, moreover, $\hat{A}[z; \alpha, \delta] \cong O(C, \hat{A}; \hat{\alpha}, \hat{\delta})$. 

Peter Kosenko (Higher School of Economics)
Arens-Michael envelopes of Ore extensions
July 11, 2017 8 / 18
Definition

Let $A$ is an Arens-Michael algebra. A family of continuous linear maps $\mathcal{T} \subset \mathcal{L}(E)$ is called $m$-localizable $\iff$ there exists a generating family of submultiplicative seminorms $\{\| \cdot \|_\nu : \nu \in \Lambda\}$ on $A$ such that for every $T \in \mathcal{T}$ and $\nu \in \Lambda$ there exists $C > 0$ such that $\|T(x)\|_\nu \leq C\|x\|_\nu$ for every $x \in A$.

Theorem (A. Pirkovskii)

Let $A$ be an algebra, $\alpha \in \text{End}(A)$ and $\delta \in \text{Der}(A, \alpha A)$, such that $(\hat{\alpha}, \hat{\delta})$ is an $m$-localizable pair of morphisms $\hat{A} \to \hat{A}$. Fix a generating family of seminorms $\{\| \cdot \|_\lambda : \lambda \in \Lambda\}$ on $\hat{A}$. Then the locally convex space

$$\mathcal{O}(C, \hat{A}) = \left\{ f = \sum_{k=0}^{\infty} a_k z^k : \sum_{k=0}^{\infty} \|a_k\|_\lambda \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\}$$
Definition

Let $A$ is an Arens-Michael algebra. A family of continuous linear maps $\mathcal{T} \subset \mathcal{L}(E)$ is called $m$-localizable $\iff$ there exists a generating family of submultiplicative seminorms $\{\| \cdot \|_\nu : \nu \in \Lambda\}$ on $A$ such that for every $T \in \mathcal{T}$ and $\nu \in \Lambda$ there exists $C > 0$ such that $\|T(x)\|_\nu \leq C\|x\|_\nu$ for every $x \in A$.

Theorem (A. Pirkovskii)

Let $A$ be an algebra, $\alpha \in \text{End}(A)$ and $\delta \in \text{Der}(A, \alpha A)$, such that $(\hat{\alpha}, \hat{\delta})$ is an $m$-localizable pair of morphisms $\hat{A} \to \hat{A}$. Fix a generating family of seminorms $\{\| \cdot \|_\lambda : \lambda \in \Lambda\}$ on $\hat{A}$. Then the locally convex space

$$\mathcal{O}(\mathbb{C}, \hat{A}) = \left\{ f = \sum_{k=0}^{\infty} a_k z^k : \sum_{k=0}^{\infty} \|a_k\|_\lambda \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\}$$

admits a unique multiplication, which is induced from $A[z; \alpha, \delta]$, such that the space $\mathcal{O}(\mathbb{C}, \hat{A})$ becomes an Arens-Michael algebra, which is denoted by $\mathcal{O}(\mathbb{C}, \hat{A}; \hat{\alpha}, \hat{\delta})$. 

Peter Kosenko (Higher School of Economics) Arens-Michael envelopes of Ore extensions July 11, 2017 8 / 18
Definition

Let $A$ is an Arens-Michael algebra. A family of continuous linear maps $\mathcal{T} \subset \mathcal{L}(E)$ is called $m$-localizable $\iff$ there exists a generating family of submultiplicative seminorms $\{\|\cdot\|_\nu : \nu \in \Lambda\}$ on $A$ such that for every $T \in \mathcal{T}$ and $\nu \in \Lambda$ there exists $C > 0$ such that $\|T(x)\|_\nu \leq C\|x\|_\nu$ for every $x \in A$.

Theorem (A. Pirkovskii)

Let $A$ be an algebra, $\alpha \in \text{End}(A)$ and $\delta \in \text{Der}(A, \alpha A)$, such that $(\hat{\alpha}, \hat{\delta})$ is an $m$-localizable pair of morphisms $\hat{A} \to \hat{A}$. Fix a generating family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on $\hat{A}$. Then the locally convex space

$$\mathcal{O}(\mathbb{C}, \hat{A}) = \left\{ f = \sum_{k=0}^{\infty} a_k z^k : \sum_{k=0}^{\infty} \|a_k\|_\lambda \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\}$$

1 admits a unique multiplication, which is induced from $A[z; \alpha, \delta]$, such that the space $\mathcal{O}(\mathbb{C}, \hat{A})$ becomes an Arens-Michael algebra, which is denoted by $\mathcal{O}(\mathbb{C}, \hat{A}; \hat{\alpha}, \hat{\delta})$,

2 and, moreover,

$$A[z; \alpha, \delta] \simeq \mathcal{O}(\mathbb{C}, \hat{A}; \hat{\alpha}, \hat{\delta}).$$
The similar theorem can be formulated for Laurent Ore extensions.
The similar theorem can be formulated for Laurent Ore extensions.

**Theorem (A. Pirkovskii)**

Let $A$ be an algebra, $\alpha \in Aut(A)$, such that $(\hat{\alpha}, \hat{\alpha}^{-1})$ is an $m$-localizable family. Fix a generating family of seminorms $\{\| \cdot \|_\lambda : \lambda \in \Lambda\}$ on $\hat{A}$. Then the locally convex space

$$O(C^*, \hat{A}) = \left\{ \sum_{k=-\infty}^{\infty} a_k z^k : \sum_{k=-\infty}^{\infty} \| a_k \|_\lambda \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\}$$

admits a unique multiplication, which is induced from $A[z, z^{-1}; \alpha]$, such that the space $O(C^*, \hat{A})$ becomes an Arens-Michael algebra, which as denoted by $O(C^*, \hat{A}; \hat{\alpha})$ and, moreover, $\hat{A}[z, z^{-1}; \alpha] \cong O(C^*, \hat{A}; \hat{\alpha})$. 

Peter Kosenko (Higher School of Economics)  
Arens-Michael envelopes of Ore extensions  
July 11, 2017  9 / 18
The similar theorem can be formulated for Laurent Ore extensions.

**Theorem (A. Pirkovskii)**

Let $A$ be an algebra, $\alpha \in \text{Aut}(A)$, such that $(\hat{\alpha}, \hat{\alpha}^{-1})$ is an $m$-localizable family. Fix a generating family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on $\hat{A}$. Then the locally convex space

$$\mathcal{O}(\mathbb{C}^*, \hat{A}) = \left\{ \sum_{k=-\infty}^{\infty} a_k z^k : \sum_{k=-\infty}^{\infty} \|a_k\|_\lambda \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\}$$

admits a unique multiplication, which is induced from $A[z, z^{-1}; \alpha]$, such that the space $\mathcal{O}(\mathbb{C}^*, \hat{A})$ becomes an Arens-Michael algebra, which as denoted by $\mathcal{O}(\mathbb{C}^*, \hat{A}; \hat{\alpha})$. 

Peter Kosenko (Higher School of Economics)
The similar theorem can be formulated for Laurent Ore extensions.

**Theorem (A. Pirkovskii)**

Let $A$ be an algebra, $\alpha \in Aut(A)$, such that $(\hat{\alpha}, \hat{\alpha}^{-1})$ is an $m$-localizable family. Fix a generating family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on $\hat{A}$. Then the locally convex space

$$
O(\mathbb{C}^*, \hat{A}) = \left\{ \sum_{k=-\infty}^{\infty} a_k z^k : \sum_{k=-\infty}^{\infty} \|a_k\|_\lambda \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\}
$$

1. admits a unique multiplication, which is induced from $A[z, z^{-1}; \alpha]$, such that the space $O(\mathbb{C}^*, \hat{A})$ becomes an Arens-Michael algebra, which as denoted by $O(\mathbb{C}^*, \hat{A}; \hat{\alpha})$

2. and, moreover,

$$
A[z, z^{-1}; \alpha] \simeq O(\mathbb{C}^*, \hat{A}; \hat{\alpha}).
$$
Definition

Suppose that $A$ is an Arens-Michael algebra and $M$ is an $A$-$\hat{\otimes}$-bimodule (the module action is jointly continuous). Fix a generating family of seminorms $\{\|\cdot\|_{\lambda} : \lambda \in \Lambda\}$ on $M$. Then we can define the following locally convex space:

$$\widehat{T}_A(M) = A \oplus \left\{ (x_k) \in \prod_{k=1}^{\infty} M^{\hat{\otimes} k} : \sum_{k=1}^{\infty} \|x_k\|_{\lambda}^{\hat{\otimes} k} \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\},$$

where

$$\|\cdot\|_{\lambda}^{\hat{\otimes} k} = \underbrace{\|\cdot\|_{\lambda} \otimes \pi \cdots \otimes \pi}_{k \text{ times}} \|\cdot\|_{\lambda}$$
Definition

Suppose that $A$ is an Arens-Michael algebra and $M$ is an $A$-$\hat{\otimes}$-bimodule (the module action is jointly continuous). Fix a generating family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on $M$. Then we can define the following locally convex space:

$$\hat{T}_A(M) = A \oplus \left\{ (x_k) \in \prod_{k=1}^{\infty} M^{\hat{\otimes} k} : \sum_{k=1}^{\infty} \|x_k\|_{\lambda}^{\hat{\otimes} k} \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\},$$

where

$$\|\cdot\|_{\lambda}^{\hat{\otimes} k} = \underbrace{\|\cdot\|_\lambda \otimes \pi \cdots \otimes \pi}_{k \text{ times}} \|\cdot\|_\lambda$$

Theorem (A. Pirkovskii)

1. $\hat{T}_A(M)$ admits a unique multiplication, which is induced from $T_A(M)$, such that $\hat{T}_A(M)$ becomes an Arens-Michael algebra.
Definition

Suppose that $A$ is an Arens-Michael algebra and $M$ is an $A$-$\hat{\otimes}$-bimodule (the module action is jointly continuous). Fix a generating family of seminorms $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ on $M$. Then we can define the following locally convex space:

$$\hat{T}_A(M) = A \oplus \left\{ (x_k) \in \prod_{k=1}^\infty M^{\otimes k} : \sum_{k=1}^\infty \|x_k\|^{\otimes k}_\lambda \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\},$$

where

$$\|\cdot\|^{\otimes k}_\lambda = \underbrace{\|\cdot\|_\lambda \otimes \pi \cdots \otimes \pi }_{k \text{ times}} \|\cdot\|_\lambda$$

Theorem (A. Pirkovskii)

1. $\hat{T}_A(M)$ admits a unique multiplication, which is induced from $T_A(M)$, such that $\hat{T}_A(M)$ becomes an Arens-Michael algebra.

2. For any algebra $A$ and $\alpha \in End(A)$ we have

$$\widehat{A[t; \alpha]} \simeq \widehat{T_A(\hat{A}_\alpha)} \simeq \hat{T}_{\hat{A}}(\hat{A}_\alpha).$$
More "explicit" description of $\hat{T}_A(A_\alpha)$

For any Arens-Michael algebra $A$ and $\alpha \in \text{End}(A)$ the algebra $\hat{T}_A(A_\alpha)$ can be represented as a non-commutative power series algebra. To do this, let us consider the canonical isomorphism

$$i_n : (A_\alpha)^\otimes_n \simeq A_\alpha^n, \quad i_n(a_1 \otimes \cdots \otimes a_n) = a_1 \alpha(a_2)\alpha^2(a_3) \cdots \alpha^{n-1}(a_n).$$
More "explicit" description of $\hat{T}_A(A_\alpha)$

For any Arens-Michael algebra $A$ and $\alpha \in End(A)$ the algebra $\hat{T}_A(A_\alpha)$ can be represented as a non-commutative power series algebra. To do this, let us consider the canonical isomorphism

$$i_n : (A_\alpha)^{\otimes n} \simeq A_\alpha^n, \quad i_n(a_1 \otimes \cdots \otimes a_n) = a_1 \alpha(a_2)\alpha^2(a_3)\cdots \alpha^{n-1}(a_n).$$

Fix a generating family of seminorms $\{|| \cdot ||_\lambda : \lambda \in \Lambda\}$ on $A$. 
More "explicit" description of $\hat{T}_A(A_\alpha)$

For any Arens-Michael algebra $A$ and $\alpha \in \text{End}(A)$ the algebra $\hat{T}_A(A_\alpha)$ can be represented as a non-commutative power series algebra. To do this, let us consider the canonical isomorphism

$$i_n : (A_\alpha)^\otimes n \simeq A_{\alpha^n}, \quad i_n(a_1 \otimes \cdots \otimes a_n) = a_1 \alpha(a_2)\alpha^2(a_3)\cdots\alpha^{n-1}(a_n).$$

Fix a generating family of seminorms $\{|| \cdot ||_\lambda : \lambda \in \Lambda\}$ on $A$.

**Proposition**

$$\hat{T}_A(A_\alpha) \simeq \left\{ \sum_{k=0}^\infty a_k x^k : \sum_{k=0}^\infty \|a_k\|^{(k)}_\lambda \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\},$$
More "explicit" description of $\hat{T}_A(A_\alpha)$

For any Arens-Michael algebra $A$ and $\alpha \in End(A)$ the algebra $\hat{T}_A(A_\alpha)$ can be represented as a non-commutative power series algebra. To do this, let us consider the canonical isomorphism

$$i_n : (A_\alpha)^n \cong A_{\alpha^n}, \quad i_n(a_1 \otimes \cdots \otimes a_n) = a_1\alpha(a_2)\alpha^2(a_3)\cdots\alpha^{n-1}(a_n).$$

Fix a generating family of seminorms $\{|| \cdot ||_\lambda : \lambda \in \Lambda\}$ on $A$.

**Proposition**

$$\hat{T}_A(A_\alpha) \cong \left\{ \sum_{k=0}^{\infty} a_k x^k : \sum_{k=0}^{\infty} ||a_k||^{(k)}_\lambda \rho^k < \infty, \lambda \in \Lambda, 0 < \rho < \infty \right\},$$

where

$$||a||^{(k)}_\lambda = \inf_{a=\sum_k i_n(a_{k1} \otimes \cdots \otimes a_{kn})} \sum_k ||a_{k1}||_\lambda \cdots ||a_{kn}||_\lambda,$$

the inf is taken w.r.t the set of all representations

$$a = \sum_{k=1}^{m} i_n(a_{k1} \otimes \cdots \otimes a_{kn}).$$
Definition

Let $A$ be an algebra and $M$ be an $A$-bimodule. Then $M$ is an invertible $A$-bimodule if there exist an $A$-bimodule $M^{-1}$ and two $A$-bimodule topological isomorphisms $i_1 : M \otimes_A M^{-1} \simeq A$ and $i_2 : M^{-1} \otimes_A M \simeq A$ (convolutions) satisfying the associativity property:
Definition

Let $A$ be an algebra and $M$ be an $A$-bimodule. Then $M$ is an invertible $A$-bimodule if there exist an $A$-bimodule $M^{-1}$ and two $A$-bimodule topological isomorphisms $i_1 : M \otimes_A M^{-1} \simeq A$ and $i_2 : M^{-1} \otimes_A M \simeq A$ (convolutions) satisfying the associativity property:

\[
\begin{align*}
M \otimes_A M^{-1} \otimes_A M & \xrightarrow{Id_M \otimes i_2} M \otimes_A A \\
A \otimes_A M & \xrightarrow{r \otimes m \mapsto rm} M \\
M^{-1} \otimes_A M \otimes_A M^{-1} & \xrightarrow{Id_M \otimes i_2} M^{-1} \otimes_A A \\
A \otimes_A M^{-1} & \xrightarrow{r \otimes n \mapsto rn} M^{-1}
\end{align*}
\]
Definition

Let $A$ be an algebra and $M$ be an $A$-bimodule. Then $M$ is an invertible $A$-bimodule if there exist an $A$-bimodule $M^{-1}$ and two $A$-bimodule topological isomorphisms $i_1 : M \otimes_A M^{-1} \simeq A$ and $i_2 : M^{-1} \otimes_A M \simeq A$ (convolutions) satisfying the associativity property:

\[
\begin{align*}
M \otimes_A M^{-1} \otimes_A M & \xrightarrow{Id_M \otimes i_2} M \otimes_A A \\
A \otimes_A M & \xrightarrow{r \otimes m \rightarrow rm} M \\
M^{-1} \otimes_A M \otimes_A M^{-1} & \xrightarrow{Id_M \otimes i_2} M^{-1} \otimes_A A \\
A \otimes_A M^{-1} & \xrightarrow{r \otimes n \rightarrow rn} M^{-1}
\end{align*}
\]

With every invertible $A$-bimodule $M$ one can associate the following algebra:

\[
L_A(M) = \bigoplus_{n \in \mathbb{Z}} M^\otimes n, \quad M^\otimes -k = (M^{-1})^\otimes k.
\]
Definition

Let $A$ be an Arens-Michel algebra and $M$ be an $A$-$\hat{\otimes}$-bimodule. Then we call $M$ an analytically invertible $A$-$\hat{\otimes}$-bimodule if there exist an $A$-$\hat{\otimes}$-bimodule $M^{-1}$ and two $A$-$\hat{\otimes}$-bimodule topological isomorphisms $i_1 : M \hat{\otimes}_A M^{-1} \simeq A$ and $i_2 : M^{-1} \hat{\otimes}_A M \simeq A$, satisfying the associativity property, as in the algebraic definition.

Example

For any Arens-Michael algebra $A$ and $\alpha \in \text{Aut}(A)$ the bimodules $A\alpha$ and $A\alpha^{-1}$ are inverse and analytically inverse $A$-$\hat{\otimes}$-bimodules.
**Definition**

Let $A$ be an Arens-Michel algebra and $M$ be an $A$-$\hat{\otimes}$-bimodule. Then we call $M$ an analytically invertible $A$-$\hat{\otimes}$-bimodule if there exist an $A$-$\hat{\otimes}$-bimodule $M^{-1}$ and two $A$-$\hat{\otimes}$-bimodule topological isomorphisms $i_1 : M \hat{\otimes}_A M^{-1} \simeq A$ and $i_2 : M^{-1} \hat{\otimes}_A M \simeq A$, satisfying the associativity property, as in the algebraic definition.

**Example**

For any Arens-Michael algebra $A$ and $\alpha \in Aut(A)$ the bimodules $A_{\alpha}$ and $A_{\alpha^{-1}}$ are inverse and analytically inverse $A$-$\hat{\otimes}$-bimodules.
Definition

Suppose that $A$ is a Fréchet-Arens-Michael algebra and $M, M^{-1}$ is a pair of analytically invertible Fréchet $A$-$\hat{\otimes}$-bimodules. Then we define

$$\hat{L}_A(M) := \hat{T}_A(M \oplus M^{-1})/I,$$

where $I$ is the smallest closed two-sided ideal, containing

$$(0, 0, x \otimes y, 0, \ldots) - (i_1(x \otimes y), 0, 0, \ldots)$$

and

$$(0, 0, y \otimes x, 0, \ldots) - (i_2(y \otimes x), 0, 0, \ldots)$$

for any $x \in M$, $y \in M^{-1}$. 

Theorem (A. Pirkovskii, P. Kosenko)

If $A$ is the algebra with Arens-Michael envelope which is Fréchet, and $\alpha \in \text{Aut}(A)$, then the following isomorphism takes place:

$$\hat{L}_A[\alpha, \alpha] \cong \hat{L}_A(A\alpha) \cong \hat{L}_{\hat{A}}(\hat{A}\hat{\alpha})$$
Definition

Suppose that $A$ is a Fréchet-Arens-Michael algebra and $M, M^{-1}$ is a pair of analytically invertible Fréchet $A$-$\hat{\otimes}$-bimodules. Then we define

$$\hat{L}_A(M) := \hat{T}_A(M \oplus M^{-1})/I,$$

where $I$ is the smallest closed two-sided ideal, containing

$$(0, 0, x \otimes y, 0, \ldots) - (i_1(x \otimes y), 0, 0, \ldots)$$

and

$$(0, 0, y \otimes x, 0, \ldots) - (i_2(y \otimes x), 0, 0, \ldots)$$

for any $x \in M, y \in M^{-1}$.

Theorem (A. Pirkovskii, P. Kosenko)

If $A$ is the algebra with Arens-Michael envelope which is Fréchet, and $\alpha \in \text{Aut}(A)$, then the following isomorphism takes place:

$$A[t, t^{-1}; \alpha] \simeq L_A(A_\alpha) \simeq \hat{L}_A(A_\alpha)$$
The quantum enveloping algebra of $\mathfrak{sl}_2$

**Definition**

The quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$ is an associative unital algebra generated by $E$, $F$, $K$, $K^{-1}$ with the following relations:

$$KE = q^2 KE, \quadKF = q^{-2} FK, \quad[E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

When $|q| = 1$, we consider the following isomorphism:

$$U_q(\mathfrak{sl}_2) \cong C[K,K^{-1}][F;\alpha_0][E;\alpha_1,\delta].$$

All morphisms turn out to be $m$-localizable, so we get the following result:

**Theorem (D. Pedchenko)**

Consider $|q| = 1, q \neq -1, 1$. Then

$$\hat{U}_q(\mathfrak{sl}_2) \cong \begin{cases} 
\sum_{i,j} c_{ijk} K^i F^j E^k : \|f\|_{\rho} := \sum |c_{ijk}| \rho^i + j + k < \infty \forall \rho > 0 
\end{cases} \quad (1)$$
The quantum enveloping algebra of $\mathfrak{sl}_2$

**Definition**

The quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$ is an associative unital algebra generated by $E$, $F$, $K$, $K^{-1}$ with the following relations:

$$KE = q^2EK, \quadKF = q^{-2}FK, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$ 

When $|q| = 1$, we consider the following isomorphism:

$$U_q(\mathfrak{sl}_2) \simeq \mathbb{C}[K, K^{-1}][F; \alpha_0][E; \alpha_1, \delta].$$

All morphisms turn out to be $m$-localizable, so we get the following result:
The quantum enveloping algebra of $\mathfrak{sl}_2$

Definition

The quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$ is an associative unital algebra generated by $E, F, K, K^{-1}$ with the following relations:

\[ KE = q^2 EK, \quad KF = q^{-2} FK, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \]

When $|q| = 1$, we consider the following isomorphism:

\[ U_q(\mathfrak{sl}_2) \cong \mathbb{C}[K, K^{-1}][F; \alpha_0][E; \alpha_1, \delta]. \]

All morphisms turn out to be $m$-localizable, so we get the following result:

Theorem (D. Pedchenko)

Consider $|q| = 1, q \neq -1, 1$. Then

\[ \overline{U_q(\mathfrak{sl}_2)} \cong \left\{ f = \sum_{i,j,k \in \mathbb{Z} \geq 0} c_{ijk} K^i F^j E^k : \| f \|_\rho := \sum_{i,j,k} |c_{ijk}| \rho^{i+j+k} < \infty \quad \forall \rho > 0 \right\} \]
When $|q| \neq 1$, the representation of $U_q(\mathfrak{sl}_2)$ as an iterated Ore extension is of no use to us because the morphisms cease to be $m$-localizable, so we need to consider another approach. Notice that

$$U_q(\mathfrak{sl}_2) \simeq \frac{\mathbb{C} \langle E, F \rangle [K, K^{-1}; \alpha]}{[E, F] - \frac{K-K^{-1}}{q-q^{-1}}}.$$ 

So it suffices to describe $\widehat{L}_R(R_\alpha)$ in the case when

$$R = \mathcal{F}_2 = \mathbb{C} \langle E, F \rangle, \alpha(E) = q^2 E, \alpha(F) = q^{-2} F.$$ 

However, we still do not know whether this algebra can be represented as an algebra consisting of (non-commutative) Laurent power series.
Examples

1. Suppose that $A = C(\mathbb{R})$ or $\mathcal{O}(\mathbb{C})$, $\alpha(f)(x) = f(x - 1)$. Then

$$\widehat{T}_A(A_\alpha) \simeq A[[x]], \quad \widehat{L}_A(A_\alpha) = 0.$$  

2. Suppose that $A = \mathcal{O}(\mathbb{C})$, and $\beta_q(f)(z) = f(qz)$. Then

$$\widehat{T}_A(A_{\beta_q}) = \left\{ f = \sum_{k \geq 0} a_k x^k : \|f\|_{\lambda, \rho} = \sum_{k \geq 0} (\|a_k\|_{\lambda|q|^{-k}}) \rho^k < \infty \quad \forall 0 < \lambda, \rho < \infty \right\}$$

when $|q| \geq 1$ and

$$\widehat{T}_A(A_{\beta_q}) = \left\{ f = \sum_{k \geq 0} a_k x^k : \|f\|_{\lambda, \rho} = \sum_{k \geq 0} (\|a_k\|_{\lambda|q|^k}) \rho^k < \infty \quad \forall 0 < \lambda, \rho < \infty \right\}$$

when $|q| \leq 1$, where $\|f\|_{\rho} = \sup_{|z| \leq \rho} |f(z)|$ or

$$\|f\|_{\rho} = \left\| \sum_{k=0}^{\infty} a_k z^k \right\|_{\rho} = \sum_{k=0}^{\infty} |a_k| \rho^k.$$
**References**

