Inverse Scattering Theory and Transmission Eigenvalues

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Scattering by an Inhomogeneous Medium

We consider the propagation of sound waves of small amplitude in $\mathbb{R}^3$ viewed as a problem in fluid dynamics. Let $v(x,t)$, $x \in \mathbb{R}^3$, be the velocity potential of a fluid particle in an inviscid fluid and

$$p(x,t) = \text{pressure}, \; \rho(x,t) = \text{density}, \; S(x,t) = \text{specific entropy}.$$

Then, if there are no external forces, we have

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v + \frac{1}{\rho} \nabla p = 0 \qquad \text{(Euler's equation)}$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \qquad \text{(equation of continuity)} \tag{1}$$

$$p = f(\rho, S) \qquad \text{(equation of state)}$$

$$\frac{\partial S}{\partial t} + v \cdot \nabla S = 0 \qquad \text{(adiabatic hypothesis)}$$

where f is a function depending on the fluid.

Scattering by an Inhomogeneous Medium

Assuming $v(x,t)$, $p(x,t)$, $\rho(x,t)$ and $S(x,t)$ are small, we perturb around the static case $v=0$, $p=p_0=$constant, $\rho=\rho_0(x)$, $S=S_0(x)$ with $p_0=f(\rho_0,S_0)$:

$$v(x,t)=\epsilon v_1(x,t)+O(\epsilon^2), \qquad p(x,t)=p_0+\epsilon p_1(x,t)+O(\epsilon^2), \tag{2}$$

$$\rho(x,t)=\rho_0(x)+\epsilon\rho_1(x,t)+O(\epsilon^2), \quad S(x,t)=S_0(x)+\epsilon S_1(x,t)+O(\epsilon^2),$$

where $0<\epsilon<<1$. Substituting (2) into (1) implies that

$$\frac{\partial v_1}{\partial t}+\frac{1}{\rho_0}\nabla p_1=0, \qquad \frac{\partial \rho_1}{\partial t}+\nabla\left(\rho_0 v_1\right)=0$$

$$\frac{\partial p_1}{\partial t}+c^2(x)\left(\frac{\partial \rho_1}{\partial t}+v_1\cdot\nabla\rho_0\right)$$

where the sound speed c is defined by

$$c^2(x)=\frac{\partial}{\partial\rho}f(\rho_0(x),S_0(x)).$$

Scattering by an Inhomogeneous Medium

We now have that

$$\frac{\partial^2 p_1}{\partial t^2} = c^2(x)\rho_0(x)\nabla\left(\frac{1}{\rho_0(x)}\nabla p_1\right).$$

If $p_1(x,t) = \mathrm{Re}\left\{u(x)e^{-i\omega t}\right\}$ we have that u satisfies

$$\rho_0(x)\nabla\left(\frac{1}{\rho_0(x)}\nabla p_1\right) + \frac{\omega^2}{c^2(x)}u = 0.$$

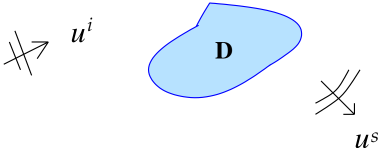
Making the further assumption that $\nabla\rho_0$ can be ignored, we arrive at

$$\Delta u + \frac{\omega^2}{c^2(x)}u = 0 \qquad (3)$$

We now assume that the slowly varying inhomogeneous medium is of compact support and is imbedded in $\mathbb{R}^3$ where the sound speed is $c(x) = c_0$ =constant.

Scattering by an Inhomogeneous Medium

We further assume that the wave motion is caused by an incident field u^i satisfying (3) with $c(x) = c_0$. We then arrive at the scattering problem of determining u such that



$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^3$$
$$u = u^s + u^i$$
$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

where $k = \omega/c_0 > 0$ is the wave number, the incident field u^i is an entire solution of the Helmholtz equation $\Delta u + k^2 u = 0$ and u^s is the scattered field.

- The function $n(x)$ is called the refractive index.
- More generally if medium is absorbing $n(x) = n_1(x) + \dfrac{n_2(x)}{k}$.
- $n(x) = 1$ outside the inhomogeneous medium D.
- $n(x)$ is piece-wise smooth, $\Re(n(x)) \geq n_0 > 0$, $\Im(n(x)) \geq 0$.

The Helmholtz Equation

We look for solutions of the Helmholtz equation

$$\Delta u + k^2 u = 0, \qquad k > 0$$

in the form

$$u(x) = f(k|x|)\, Y_n^m(\hat{x})$$

where $x \in \mathbb{R}^3$, $\hat{x} = x/|x|$, Y_n^m is a spherical harmonic

$$Y_n^m(\theta, \varphi) := \left(\frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!} \right)^{1/2} P_n^m(\cos\theta) e^{im\varphi}$$

$m = -n, \cdots, n$, $n = 0, 1, 2, \cdots$, (θ, φ) are the spherical angles of $\hat{x}$ and P_n^m is an associated Legendre polynomial.

Note that $\{Y_n^m\}$ is a complete orthonormal system in $L^2(S^2)$ where $Y_0^0 = \dfrac{1}{\sqrt{4\pi}}$ and

$$S^2 := \{x : |x| = 1\}.$$

The Helmholtz Equation

The function f is a solution of the spherical Bessel equation

$$t^2 f''(t) + 2tf'(t) + \left[t^2 - n(n+1)\right] f(t) = 0$$

with two linearly independent solutions

$$j_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p t^{n+2p}}{2^p p! 1 \cdot 3 \cdots (2n+2p+1)}$$

$$y_n(t) := \frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p-n-1}}{2^p p! (-2n+1)(-2n+3) \cdots (-2n+2p-1)}$$

called, respectively, the spherical Bessel function and the spherical Neumann function. The functions

$$h_n^{(1)}(t) := j_n(t) + i y_n(t) \qquad h_n^{(2)}(t) := j_n(t) - i y_n(t)$$

are called spherical Hankel functions of order n.

The Helmholtz Equation

From the series representation of j_n and y_n we have that for $f_n = j_n$ or $f_n = y_n$ we have that

$$f_{n+1}(t) = -t^n \frac{d}{dt}\left\{t^{-n} f_n(t)\right\}$$

for $n = 0, 1, 2, \cdots$ and

$$h_0^{(1,2)}(t) = \frac{e^{\pm it}}{\mp it}.$$

From this we see that the spherical Hankel functions have the asymptotic behavior

$$h_n^{(1,2)}(t) = \frac{1}{t} e^{\pm i\left(t - \frac{n\pi}{2} - \frac{\pi}{2}\right)} \left\{1 + O\left(\frac{1}{t}\right)\right\}$$

as $t \to \infty$. In particular, $h_n^{(1)}(kr)$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0,$$

i.e. if $u(x) = h_n^{(1)}(kr) Y_n^m(\hat{x})$ then $u(x)e^{-i\omega t}$ (where ω is the frequency and t is time) is an outgoing wave.

The Helmholtz Equation

Solutions of the Helmholtz equation satisfying the Sommerfeld radiation condition uniformly in $\hat{x}$ are called radiating.

Now let D be a bounded domain such that $\mathbb{R}^3 \setminus \overline{D}$ is connected and assume that ∂D is of class C^2 with unit normal ν directed into the exterior of D. Let

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \qquad x \neq y$$

be the radiating fundamental solution to the Helmholtz equation.

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|} \left\{ e^{-ik|\hat{x}|\cdot y} + O\left(\frac{1}{|x|}\right) \right\}$$

$$\frac{\partial}{\partial \nu_y} \frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|} \left\{ \frac{\partial}{\partial \nu_y} e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right) \right\}$$

as $|x| \to \infty$, $\hat{x} = x/|x|$ uniformly for all $y \in \partial D$.

The Helmholtz Equation

Using Green's second identity

$$\int_D (u\Delta v - v\Delta u)\, dx = \int_D \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) ds$$

we can deduce Green's formula for functions $u \in C^2(D) \cap C^1(\overline{D})$:

$$u(x) = \int_{\partial D} \left\{ \frac{\partial u}{\partial \nu}\Phi(x,y) - u\frac{\partial}{\nu_y}\Phi(x,y) \right\} ds$$

$$- \int_D \left\{ \Delta u + k^2 u \right\} \Phi(x,y) dy, \quad x \in D.$$

Theorem

Let $u \in C^2(D) \cap C^1(\overline{D})$ be a solution to the Helmholtz equation in D. Then u is real analytic in D.

The Helmholtz Equation

Holmgren's Theorem

Let $u \in C^2(D) \cap C^1(\overline{D})$ be a solution to the Helmholtz equation in D such that

$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \;\; \Gamma$$

for some open subset $\Gamma \subset \partial D$. Then u is identically zero in D.

For x in the exterior of D we have the following theorem:

Theorem

Let $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ be a radiating solution to

$$\text{the Helmholtz equation} \qquad \Delta u + k^2 u = 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}.$$

Then we have Green's formula

$$u(x) = \int_{\partial D} \left\{ u \frac{\partial}{\nu_y} \Phi(x,y) - \frac{\partial u}{\partial \nu_y} \Phi(x,y) \right\} ds, \qquad x \in \mathbb{R}^3 \setminus \overline{D}.$$

The Helmholtz Equation

Corollary

An entire solution to the Helmholtz equation satisfying the radiation condition must vanish identically.

Corollary

Every radiating solution to the Helmholtz equation has the asymptotic behavior of an outgoing spherical wave

$$u(x) = \frac{e^{ik|x|}}{|x|} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \qquad |x| \to \infty$$

uniformly in all directions $\hat{x} = x/|x|$.

The function u_∞ defined on the unit sphere S^2 is called the far field pattern of u.

The Helmholtz Equation

Rellich's Lemma

Let $u \in C^2(\mathbb{R}^3 \setminus \overline{D})$ be a solution to the Helmholtz equation satisfying

$$\lim_{r \to \infty} \int_{|x|=r} |u(x)|^2 \, dx = 0.$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.

Corollary

Assume $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus \overline{D})$ is a radiating solution to the Helmholtz equation such that

$$\operatorname{Im} \int_{\partial D} u \frac{\partial \overline{u}}{\partial \nu} \, ds = 0.$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.

Scattering by an Inhomogeneous Medium

Let us set $m := 1 - n$. Hence

$$D := \left\{x \in \mathbb{R}^3 : \; m(x) \neq 0\right\}.$$

We again let

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \qquad x \neq y.$$

Theorem

Given two bounded domains D and G, the volume potential

$$(V\varphi)(x) := \int_D \Phi(x, y)\varphi(y) dy, \qquad x \in \mathbb{R}^3$$

defines a bounded operator $V : L^2(D) \to H^2(G)$.

Scattering by an Inhomogeneous Medium

We now show that the scattering problem (4) $-$ (6) is equivalent to solving the Lippmann-Schwinger integral equation

$$u(x) = u^i(x) - k^2 \int_{\mathbb{R}^3} \Phi(x,y) m(y) u(y)\, dy, \qquad x \in \mathbb{R}^3. \tag{7}$$

Theorem

If $u \in H^2_{loc}(\mathbb{R}^3)$ is a solution of (4) $-$ (6) then u is a solution of (7). Conversely, if $u \in C(\mathbb{R}^3)$ is a solution of (7) then $u \in H^2_{loc}(\mathbb{R}^3)$ and u is a solution of (4) $-$ (6).

Theorem

Suppose that $m(x) = 0$ for $|x| \geq a$ and $k^2 < 2/Ma^2$ where $M = \sup_{|x| \leq a} |m(x)|$. Then there exists a unique solution to the Lippmann-Schwinger integral equation

Scattering by an Inhomogeneous Medium

From (7) we see that

$$u^s(x) = -k^2 \int_{\mathbb{R}^3} \Phi(x,y)m(y)u(y)\,dy, \qquad x \in \mathbb{R}^3$$

and hence

$$u^s(x) = \frac{e^{ik|x|}}{|x|} u_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \qquad |x| \to \infty$$

where the far field pattern u_∞ is given by

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y)u(y)\,dy, \qquad \hat{x} = \frac{x}{|x|}.$$

Assuming k is sufficiently small and replacing u by the first term in solving (7) by iteration (the weak scattering assumption) gives the Born approximation

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y)u^i(y)\,dy.$$

Scattering by an Inhomogeneous Medium

Unique Continuation Principle

Let G be a domain in $\mathbb{R}^3$ and suppose $u \in H^2(G)$ is a solution of

$$\Delta u + k^2 n(x) u = 0$$

in G such that n is piecewise continuous in G and u vanishes in a neighborhood of some $x_0 \in G$. Then u is identically zero in G.

Theorem

For each $k > 0$ there exists a unique solution $u \in H^2_{loc}(\mathbb{R}^3)$ to the scattering problem $(4) - (6)$ and u depends continuously with respect to the maximum norm on the incident field u^i.