Inverse Scattering Theory and Transmission Eigenvalues

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The Far Field Operator

Now let $u^i(x) = e^{ikx\cdot d}$, $|d| = 1$, and consider the scattering problem

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^3 \tag{1}$$

$$u(x) = u^i(x) + u^s(x) \tag{2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{3}$$

with corresponding far field pattern $u_\infty(\hat{x}) = u_\infty(\hat{x}, d)$ defined by

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|}\right)\right\}.$$

Reciprocity Principle

Let $u_\infty(\hat{x}, d)$ be the far field pattern corresponding to $(1) - (3)$. Then

$$u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x}).$$

The Far Field Operator

Proof: Let $D \subset \{x : |x| < a\}$ where $D := \{x : m(x) \neq 0\}$. We now use Green's second identity to obtain

$$0 = \int\limits_{|y|=a} \left\{ u^i(y,d) \frac{\partial}{\partial \nu} u^i(y,-\hat{x}) - u^i(y,-\hat{x}) \frac{\partial}{\partial \nu} u^i(y,d) \right\} ds(y)$$

$$0 = \int\limits_{|y|=a} \left\{ u^s(y,d) \frac{\partial}{\partial \nu} u^s(y,-\hat{x}) - u^s(y,-\hat{x}) \frac{\partial}{\partial \nu} u^s(y,d) \right\} ds(y).$$

The second corollary to Green's formula in Lecture 2 shows that

$$4\pi u_\infty(\hat{x},d) = \int\limits_{|y|=a} \left\{ u^s(y,d) \frac{\partial}{\partial \nu} u^i(y,-\hat{x}) - u^i(y,-\hat{x}) \frac{\partial}{\partial \nu} u^s(y,d) \right\} ds(y)$$

$$4\pi u_\infty(-d,-\hat{x}) = \int\limits_{|y|=a} \left\{ u^s(y,-\hat{x}) \frac{\partial}{\partial \nu} u^i(y,d) - u^i(y,d) \frac{\partial}{\partial \nu} u^s(y,-\hat{x}) \right\} ds(y).$$

The Far Field Operator

Subtracting the last of the above equations from the sum of the first three gives

$$4\pi\left[u_\infty(\hat{x}, d) - u_\infty(-d, -\hat{x})\right]$$

$$= \int\limits_{|y|=a} \left\{ u(y, d)\frac{\partial}{\partial\nu}u(y, -\hat{x}) - u(y, -\hat{x})\frac{\partial}{\partial\nu}u(y, d) \right\} ds(y)$$

$$= 0$$

by Green's second identity.

The Far Field Operator

We now define the far field operator $F : L^2(S^2) \to L^2(S^2)$ by

$$(Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d) g(d) ds(d).$$

Since $u_\infty(\hat{x}, d)$ is infinitely differentiable with respect to each of its variables, F is clearly compact.

The corresponding scattering operator $S : L^2(S^2) \to L^2(S^2)$ is defined by

$$S := I + \frac{ik}{4\pi} F.$$

The Far Field Operator

Lemma

For $g, h \in L^2(S^2)$ define the Herglotz wave functions v^i and w^i with kernels g and h respectively by

$$v^i(x) := \int_{S^2} e^{ikx\cdot d} g(d) ds(d) \qquad w^i(x) := \int_{S^2} e^{ikx\cdot d} h(d) ds(d).$$

Let v and w be the solutions of the scattering problem $(1) - (3)$ corresponding to the incident fields v^i and w^i respectively. Then

$$ik^2 \int_D \Im(n) v\overline{w}\, dx = 2\pi\,(Fg, h) - 2\pi\,(g, Fh) - ik\,(Fg, Fh)\,.$$

Theorem

If $\Im(n) = 0$, then the far field operator f is normal, i.e. $F^*F = FF^*$, and the scattering operator S is unitary, i.e. $SS^* = S^*S = I$.

The Far Field Operator

We now introduce the transmission eigenvalue problem: Determine $k > 0$ and $v, w \in L^2(D)$, $v - w \in H_0^2(D)$ such that v and w are not identically zero and

$$\begin{aligned} \Delta w + k^2 n w &= 0 && \text{in} \quad D \\ \Delta v + k^2 v &= 0 && \text{in} \quad D \\ w &= v && \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} && \text{on} \quad \partial D \end{aligned}$$

Such values of k are called transmission eigenvalues.
Recall that $D := \{x : n(x) \neq 1\}$ and it is assumed that D is bounded with C^2 boundary ∂D such that $\mathbb{R}^3 \setminus \overline{D}$ is connected.

Theorem

Let F be the far field operator corresponding to the scattering problem $(1) - (3)$. Then F is injective if k is not a transmission eigenvalue

The Far Field Operator

Proof: Suppose $Fg = 0$. Then the far field pattern w_∞ of the scattered field w^s corresponding to the incident field

$$w^i(x) = \int_{S^2} e^{ikx\cdot d} g(d)ds(d)$$

vanishes. By Rellich's lemma $w^s = w - w^i$ vanishes outside D. Then $w = w^s + w^i$ satisfies $\Delta w + k^2 n w = 0$ in $\mathbb{R}^3$ and $w - w^i = 0$ on ∂D, $\frac{\partial}{\partial \nu}(w - w^i) = 0$ on ∂D. If k is not a transmission eigenvalue then $w^i = w = 0$ which implies $g = 0$, i.e. F is injective.

Corollary

Let F be the far field operator corresponding to the scattering problem (1) – (3). Then F has dense range if k is not a transmission eigenvalue.

The Far Field Operator

Proof: From $(Fg, h) = (g, F^*h)$ we have $R(F)^\perp = N(F^*)$.
Hence we must show that $F^*h = 0 \implies h = 0$.

To this end, using reciprocity, we have that $F^*h = 0$

$$\implies \int_{S^2} \overline{u_\infty(d, \hat{x})} h(d) ds(d) = 0$$

$$\implies \int_{S^2} u_\infty(-\hat{x}, -d) \overline{h(d)} ds(d) = 0$$

$$\implies \int_{S^2} u_\infty(\hat{x}, d) \overline{h(-d)} ds(d) = 0$$

$$\implies h = 0 \quad \text{by the previous theorem.}$$

Inverse Scattering

We again consider the scattering problem

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^3 \tag{1}$$

$$u = u^i + u^s \tag{2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \tag{3}$$

It has previously been shown that

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}, d) + O\left(\frac{1}{|x|}\right)\right\}, \qquad |x| \to \infty$$

where

$$u_\infty(\hat{x}) = -\frac{k^2}{4\pi}\int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y)u(y)\,dy.$$

Our aim is to determine $n(x)$ from $u_\infty(\hat{x}, d)$. We begin with uniqueness.

Inverse Scattering

Theorem

The set of products $h_1 h_2$ of entire harmonic functions h_1 and h_2 is complete in $L^2(D)$ for any bounded domain $D \subset \mathbb{R}^3$.

Proof: Given $y \in \mathbb{R}^3$ choose a vector $b \in \mathbb{R}^3$ with $b \cdot y = 0$ and $|b| = |y|$. Then for $z := y + ib \in \mathbb{C}^3$ we have $z \cdot z = 0 \Longrightarrow$ $h_z(x) := e^{iz \cdot x}$, $x \in \mathbb{R}^3$, is harmonic.

Now assume $\varphi \in L^2(D)$ is such that

$$\int_D \varphi h_1 h_2 \, dx = 0$$

for all pairs of entire harmonic functions h_1 and h_2. For $h_1 = h_z$, $h_2 = h_{\bar{z}}$ we have that

$$\int_D \varphi e^{2iy \cdot x} dx = 0$$

for $y \in \mathbb{R}^3 \Longrightarrow \varphi = 0$ a.e. by the Fourier integral theorem.

Inverse Scattering

To prove uniqueness for the inverse scattering problem of determining $n(x)$ from $u_\infty(\hat{x}, d)$ we need a property corresponding to the above theorem for products $v_1 v_2$ of solutions to

$$\Delta v_1 + k^2 n_1 v_1 = 0 \qquad \text{and} \qquad \Delta v_2 + k^2 n_2 v_2 = 0$$

for two different refractive indices n_1 and n_2. Such a result was first established by SYLVESTER AND UHLMANN:

Theorem

Let B be an open ball centered at the origin and containing the support of $m := 1 - n$. Then there exists a constant $c > 0$ such that for each $z \in \mathbb{C}^3$ with $z \cdot z = 0$ and $|\Re(z)| > 2k^2 \|n\|_\infty$ there exists a solution $v \in H^2(D)$ of $\Delta v + k^2 n v = 0$ in B of the form

$$v(x) = e^{iz\cdot x}\,[1 + w(x)]$$

where

$$\|w\|_{L^2(D)} \leq \frac{c}{\mathrm{Re}(z)}.$$

Inverse Scattering

Theorem

Let B and B_0 be two open balls centered at the origin and containing the support of $m = 1 - n$ such that $\overline{B} \subset B_0$. Then the set of total fields $\{u(\cdot, d) : d \in S^2\}$ satisfying (1) – (3) is complete in the closure of

$$\{v \in H^2(D) : \Delta v + k^2 n v = 0, \text{ in } B_0\}$$

with respect to the $L^2(B)$ norm.

Theorem

The index of refraction n is uniquely determined by a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in S^2$ and a fixed wave number k.

Inverse Scattering

Born or Weak Scattering Approximation

It is based on linearized model

$$u_\infty(\hat{x}, d) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y) u^i(y, d)\, dy.$$

Nonlinear Optimization Methods

$$\mathcal{F} : m \longmapsto u_\infty(\hat{x}, d).$$

Letting B be a ball containing the (unknown) support of m, we interpret $\mathcal{F}$ as an operator from $L^2(B)$ into $L^2(S^2 \times S^2)$.

From the Lippman-Schwinger equation we can write

$$(\mathcal{F}m)(\hat{x}, d) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} m(y) u(y)\, dy \tag{4}$$

where $u(\cdot, d)$ is the unique solution of

$$u(x, d) + k^2 \int_{\mathbb{R}^3} \Phi(x, y) m(y) u(y, d)\, dy = u^i(x, d), \; x \in B \tag{5}$$

Recall now that a mapping $\mathcal{T} : X \to Y$ of a normed space X into a normed space Y is called Fréchet differentiable if there exists a bounded linear operator $\mathcal{A} : X \to Y$ such that

$$\lim_{h \to 0} \frac{1}{\|h\|} \|\mathcal{T}(x+h) - \mathcal{T}(x) - \mathcal{A}h\| = 0$$

and we write $\mathcal{T}'(x) := \mathcal{A}$.

In particular, from (5) it can be seen that the Fréchet derivative $v := u'_m h$ of u with respect to m (in the “direction” h) satisfies the Lippman-Schwinger equation

$$v(x,d) + k^2 \int_{\mathbb{R}^3} \Phi(x,y) \left[m(y) v(y,d) + h(y) u(y,d) \right] dy = 0, \ x \in B. \quad (6)$$

Inverse Scattering

From (4) we have that

$$(\mathcal{F}'_m h)(\hat{x}, d) = -\frac{k^2}{4\pi} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} \left[m(y)v(y,d) + h(y)u(y,d)\right] \, dy$$

i.e. $(\mathcal{F}'_m h)(\hat{x})$ coincides with the far field pattern of the solution $v(\cdot, d) \in H^2_{loc}(\mathbb{R}^3)$ of (6). Note that $\mathcal{F}'_m : L^2(B) \to L^2(S^2 \times S^2)$ is compact.

Theorem

The operator $\mathcal{F} : m \longmapsto u_\infty$ is Fréchet differentiable. The derivative is given by

$$\mathcal{F}'_m h = v_\infty$$

where v_∞ is the far field pattern of the radiating solution $v \in H^2_{loc}(\mathbb{R}^3)$ to

$$\Delta v + k^2 n v = -k^2 u h \qquad \text{in } \mathbb{R}^3.$$

Inverse Scattering

Theorem

The operator $\mathcal{F}'_m : L^2(B) \to L^2(S^2 \times S^2)$ is injective.

Using the previous two theorems we can now apply Newton's method to the non-linear equation

$$\mathcal{F}(m) = u_\infty.$$

However to implement this procedure we must solve a direct scattering problem at each step of the iteration procedure. We furthermore have the problem of local minima and need to solve an "ill-posed" operator equation of the first kind at each step.