Inverse Scattering Theory and Transmission Eigenvalues

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Ill-Posed Problems

Definition

Let $A : X \to V \subset Y$ be an operator from a normed space X into a subset V of a normed space Y. The equation

$$A\varphi = f$$

is called well-posed if $A : X \to V$ is bijective and the inverse operator $A^{-1} : V \to X$ is continuous. Otherwise the equation is ill-posed.

Theorem

Let A be a linear compact operator from $X \to V \subset Y$.
Then $A\varphi = f$ is ill-posed if X is not of finite dimension.

Proof. If $A^{-1} : V \to X$ exists and is continuous then $I = A^{-1}A$ is compact which implies that X is finite dimensional.

Ill-Posed Problems

We now assume that A is a linear compact operator and wish to approximate the solution φ to $A\varphi = f$ from a knowledge of a perturbed right hand side f^δ with a known error level $\|f^\delta - f\| \leq \delta$.

We will always assume that $A : X \to Y$ is injective and want the approximate solution φ^δ to depend continuously on f^δ.

Definition

Let $A : X \to Y$ be an injective compact linear operator. A family of bounded linear operators $R_\alpha : Y \to X$ with the property that

$$R_\alpha \to A^{-1} f, \qquad \alpha \to 0 \tag{1}$$

for all $f \in A(X)$ is called a regularization scheme for A. The parameter α is called the regularization parameter.

Ill-Posed Problems

It is easily verified that if dim $X = \infty$ the operators R_α can not be uniformly bounded with respect to α and the operators $R_\alpha A$ cannot be norm convergent as $\alpha \to 0$.

The regularization scheme approximates the solution φ of $A\varphi = f$ by the regularized solution

$$\varphi_\alpha^\delta := R_\alpha f^\delta.$$

Hence $$\varphi_\alpha^\delta - \varphi = R_\alpha f^\delta - R_\alpha f + R_\alpha A\varphi - \varphi$$

$$\|\varphi_\alpha^\delta - \varphi\| \leq \delta \|R_\alpha\| + \|R_\alpha A\varphi - \varphi\|.$$

The error consists of two parts: the first term reflects the error in the data and the second part the error between R_α and A^{-1}. The first term will be increasing as $\alpha \to 0$ due to the ill-posed nature of the problem whereas the second term will be decreasing as $\alpha \to 0$

A natural strategy is the Morozov discrepancy principle which is based on the idea that the residual should not be smaller than the accuracy of the measurements, i.e α should be chosen such that

$$\|AR_\alpha f^\delta - f^\delta\| = \gamma\delta$$

for some parameter $\gamma \geq 1$.

Now let $A : X \to Y$ be a compact linear operator with adjoint $A^* : Y \to X$. The nonnegative square roots of the eigenvalues of $A^*A : X \to X$ are called the singular values of A. We assume from now on that $A \neq 0$ and that X and Y are Hilbert spaces.

Theorem

Let (μ_n), $\mu_1 \geq \mu_2 \geq \cdots$ be the singular values of A. Then there exist orthonormal sequences (φ_n) in X and (g_n) in Y such that

$$A\varphi_n = \mu_n g_n, \qquad A^* g_n = \mu_n \varphi_n$$

and for all $\varphi \in X$

$$\varphi = \sum_{n=1}^{\infty} (\varphi, \varphi_n)\, \varphi_n + Q\varphi$$

$$A\varphi = \sum_{n=1}^{\infty} \mu_n\, (\varphi, \varphi_n)\, g_n$$

where $Q : X \to N(A)$ is the orthogonal projection operator. The system (μ_n, φ_n, g_n) is called a singular system of A.

Ill-Posed Problems

Let $A : X \to Y$ *be a compact linear operator with singular system* (μ_n, φ_n, g_n)*. Then*

$$A\varphi = f$$

is solvable if and only if $f \in N(A^*)^\perp$ *and satisfies*

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} |(f, g_n)|^2 < \infty. \qquad (2)$$

In this case a solution is given by

$$\varphi = \sum_{n=1}^{\infty} \frac{1}{\mu_n} (f, g_n) \varphi_n. \qquad (3)$$

Ill-Posed Problems

The regularized solution

$$\varphi_\alpha = \sum_{n=1}^{\infty} \frac{1}{\mu_n} q(\alpha, \mu)\, (f, g_n)\, \varphi_n$$

with $q(\alpha, \mu) = \frac{\mu^2}{\alpha + \mu^2}$ leads to Tikhonov regularization.

Theorem

Let $A : X \to Y$ be a compact linear operator. Then for each $\alpha > 0$ the operator $\alpha I + A^* A : X \to X$ is bijective and has bounded inverse. Furthermore, if A is injective then

$$R_\alpha := (\alpha I + A^* A)^{-1} A^*$$

describes a regularization scheme with $\|R_\alpha\| \leq \dfrac{1}{2\sqrt{\alpha}}$.

Qualitative Methods

An alternative approach to weak scattering methods and nonlinear optimization techniques are qualitative methods. Such methods seek partial information such as the support D and estimates on n

- Linear sampling method (COLTON-KIRSCH (1996)) ... and Factorization method (KIRSCH (1998)) ...
- Singular sources method (POTTHAST (2001)) ...
- Convex scattering support (KUSIAK-SYLVESTER (2003), GRIESMAIER-HANKE-SYLVESTER (2013)) ...
- ...

CAKONI-COLTON (2014), *A Qualitative Approach to Inverse Scattering Theory, Springer*.

A. KIRSCH AND N. GRINBERG (2008), *The Factorization Method for Inverse Problems, Oxford University Press*.

Linear Sampling Method

The linear sampling method is based on solving the far field equation

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z, k), \quad \text{for} \quad g \in L^2(S^2), \quad \hat{x} \in S^2,\ z \in \mathbb{R}^m$$

for a fixed k, where $\Phi_\infty(\hat{x}, z, k) := e^{-ik\hat{x}\cdot z}$.

If k is not a transmission eigenvalue, then there is a $g_\epsilon := g_{z,k,\epsilon}$ satisfying

$$\|Fg_\epsilon - \Phi_\infty(\cdot, z, k)\|_{L^2(S^{\copyright})} < \epsilon$$

such that

- for $z \in D$ $\quad \lim_{\epsilon\to 0} \|v_{g_\epsilon}\|_{L^2(D)}$ exists
- for $z \notin D$ $\quad \lim_{\epsilon\to 0} \|v_{g_\epsilon}\|_{L^2(D)} = \infty$

where $v_g(x) := \int_{S^2} e^{ikx\cdot d} g(d)\, ds(d)$

The Linear Sampling Method

Two main ingredients:

- If k is not a transmission eigenvalue than there exits a unique solution $v, w \in L^2(D)$, $v - w \in H^2(D)$, of the inhomogeneous interior transmission problem

$$\Delta w + k^2 n w = 0 \quad \text{in} \quad D \tag{1}$$
$$\Delta v + k^2 v = 0 \quad \text{in} \quad D \tag{2}$$
$$w - v = \Phi(\cdot, z) \quad \text{on} \quad \partial D \tag{3}$$
$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu}\Phi(\cdot, z) \quad \text{on} \quad \partial D. \tag{4}$$

- The set $\{v_g : \; g \in L^2(S^2)\}$ is dense in

$$\left\{u \in L^2(D) : \;\; \Delta u + k^2 u = 0 \text{ in } D\right\}.$$

The Linear Sampling Method

A problem with the linear sampling method is that, in general, there does not exist a solution to

$$Fg = \Phi_\infty(\cdot, z)$$

for noise free data and hence it is not clear what solution is obtained by using Tikhonov regularization to solve this equation.

In particular, it is not clear whether Tikhonov regularization indeed leads to the approximations predicted by the above theorem.

The problem can be resolved by using the theory of the factorization method.

The Factorization Method

To fix our ideas, we assume that $\Im(n) = 0$ (i.e. the far field operator is normal) and $n(x) > 1$.

We rewrite the scattering problem as

$$\Delta u^s + k^2 n u^s = k^2 m\, e^{ikx\cdot d} \quad \text{in } \mathbb{R}^3$$

and u^s satisfies the Sommerfeld radiation condition

More generally, for $f \in L^2(D)$ we consider

$$\Delta w + k^2 n w = mf \quad \text{in } \mathbb{R}^3$$

and w satisfies the Sommerfeld radiation condition, which has a unique solution $w \in H^2_{loc}(\mathbb{R}^3)$.

- Define the $G : L^2(D) \to L^2(S^2)$

$$f \to w_\infty$$

The Factorization Method

Factorization of the far field operator

Theorem

Let F and G be defined as above. Then

$$F = 4\pi k^2 G S^* G^*$$

where S^* is the adjoint of $S : L^2(D) \to L^2(D)$ defined by

$$(S\psi)(x) := -\frac{\psi(x)}{m(x)} - k^2 \int_D \Phi(x,y)\psi(y)dy, \qquad x \in D$$

and $G^* : L^2(S^2) \to L^2(D)$ is the adjoint of G.

Proof: We have that $u_\infty = k^2 G u^i$. We now define the Herglotz operator $H : L^2(S^2) \to L^2(D)$ by

$$(Hg)(x) := \int_{S^2} e^{ikx\cdot d} g(d) ds(d), \qquad x \in D,$$

and note that Fg is the far field pattern corresponding to the incident field Hg, i.e.

$$F = k^2 GH.$$

Since

$$(H^*\psi)(\hat{x}) := \int_D e^{-ik\hat{x}\cdot y} \psi(y) dy, \qquad \hat{x} \in S^2$$

we have that $H^*\psi = 4\pi w_\infty$ where w_∞ is the far field pattern of

$$w(x) := \int_D \Phi(x, y) \psi(y) dy, \qquad x \in \mathbb{R}^3.$$

The Factorization Method

But

$$\Delta w + k^2 n w = -m\left(\frac{\psi}{m} + k^2 w\right)$$

and hence

$$H^*\psi = 4\pi w_\infty = -4\pi G\left(\frac{\psi}{m} + k^2 w\right) = 4\pi G S\psi$$

i.e. $H^* = 4\pi GS$. Thus $H = 4\pi S^* G^*$ and since $F = k^2 GH$ the theorem follows.

Lemma

For $z \in \mathbb{R}^3$ let

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x}\cdot z}$$

be the far field pattern of $\Phi(x, z)$.

Then $z \in D$ if and only if $\Phi_\infty(\cdot, z)$ is in the range of G.

The Factorization Method

Proof: For $z \in D$ choose a cut-off function $\rho \in C^\infty(\mathbb{R}^3)$ which vanishes near z and equals one in $\mathbb{R}^3 \setminus D$. Then $v(x) = \rho(x)\Phi(x, z)$ has $\Phi_\infty(\hat{x}, z)$ as its far field pattern. Hence

$$\Phi_\infty(\cdot, z) = Gf$$

where

$$f := \frac{1}{m}\left(\Delta v + k^2 n v\right) \in L^2(D).$$

Now assume that $z \notin D$ and $\Phi_\infty(\cdot, z) = Gf$ for some $f \in L^2(D)$. By Rellich's lemma $\Phi(\cdot, z) = u^s$ in the exterior of $D \cup \{z\}$ and this is a contradiction since u^s is smooth near z but $\Phi(\cdot, z)$ is singular at z.

The Factorization Method

Theorem

Let X and H be Hilbert spaces and assume that $F : H \to H$, $B : X \to H$ and $T : X \to X$ are bounded linear operators that satisfy

$$F = BTB^* \qquad \text{where } B^* \text{ is the adjoint of } B$$

$$\operatorname{Im}(Tf, f) \neq 0 \qquad \text{for all } f \in \overline{B^*(H)} \quad \text{with } f \neq 0$$

and $T = T_0 + C$ where C is compact such that

$$(T_0 f, f) \in \mathbb{R}, \quad (T_0 f, f) \geq c\|f\|^2 \quad \text{for all } f \in \overline{B^*(H)} \text{ and some } c > 0$$

In addition let the operator F be compact, injective and assume that $I + i\gamma F$ is unitary for some $\gamma > 0$.

Then the ranges $B(X)$ and $(F^*F)^{1/4}(H)$ coincide.

The Factorization Method

Theorem

Let $S : L^2(D) \to L^2(D)$ be defined as above and let $S_0 : L^2(D) \to L^2(D)$ be given by

$$S_0\psi := -\frac{\psi}{m}.$$

Then

1. S_0 is bounded, self-adjoint and satisfies

$$(S_0\psi, \psi) \geq \frac{1}{\|m\|_\infty}\|\psi\|^2, \qquad \psi \in L^2(D).$$

2. $S - S_0 : L^2(D) \to L^2(D)$ is compact.
3. S is an isomorphism from $L^2(D)$ onto $L^2(D)$.
4. $\operatorname{Im}(S\psi, \psi) \leq 0$ for all $\psi \in L^2(D)$ with strict inequality holding for all $\psi \in \overline{G^*(L^2(D))}$ with $\psi \neq 0$.

Proof:

1. This follows since $m(x) > 0$ for $x \in \overline{D}$.
2. This follows from the fact that volume potentials map $L^2(D) \to H^2(D)$ and $H^2(D)$ is compactly embedded into $L^2(D)$.
3. From its definition, S_0 clearly has a bounded inverse. Hence by 2. and the Riesz-Fredholm theory it suffices to show that S is injective. Suppose $S\psi = 0$. Then $\varphi = \dfrac{\psi}{m}$ satisfies the homogeneous Lippmann-Schwinger equation and hence $\varphi = 0$ and then $\psi = 0$.
4. Proof is technical - not given.

The Factorization Method

Theorem

Assume that $n(x) > 1$ for $x \in \overline{D}$ and $k > 0$ is not a transmission eigenvalue.

Then $z \in D$ if and only if $\Phi_\infty(\cdot, z)$ is in the range of $(F^*F)^{1/4}$.

The theorem is also true if we assume that $0 < n(x) < 1$ for $x \in \overline{D}$.

To construct D, let λ_n and ψ_n be the eigenvalues and eigenfunctions of F. Then $\left(\sqrt{|\lambda_n|}, \psi_n, \psi_n\right)$ is a singular system for $(F^*F)^{1/4}$ and by Picard's theorem

$$z \in D \Longleftrightarrow \sum_{n=1}^{\infty} \frac{|(\psi_n, \Phi_\infty(\cdot, z))|^2}{|\lambda_n|} < \infty.$$

Linear Sampling Method

The factorization method can be used to provide the following justification of the linear sampling method.

ARENS-LECHLEITER (2009) - *J. Int. Eqns and Appl.*

Assume k is not a transmission eigenvalue. Consider

$$(F^*F)^{1/4}\varphi_z = \Phi_\infty(\cdot, z, k) \qquad \text{and} \qquad (\alpha I + F^*F)g_{z,k,\alpha} = F^*\Phi_\infty(\cdot, z)$$

Then $c\|\varphi_z\|^2 \leq \lim_{\alpha\to 0}|v_{g_\alpha^z}(z)| \leq C\|\varphi_z\|^2$, i.e.

- for $z \in D$ $\quad \lim_{\alpha\to 0}|v_{g_{z,k,\alpha}}(z)|$ exists
- for $z \notin D$ $\quad \lim_{\alpha\to 0}|v_{g_{z,k,\alpha}}(z)| = \infty$

Hence to implement the linear sampling method one can plot $|v_{g_{z,k,\alpha}}(z)|$ or (as done in practice) $\|g_{z,k,\alpha}\|_{L^2(\Omega)}$