Fourier series and the discrete Fourier transform

802647S

Lecture Notes

1^{st} Edition

Fourth printing

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2014

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1 Preliminaries

Definition 1.1. A function $f(x)$ of one variable x is said to be *periodic* with period $T > 0$ if the domain $D(f)$ of f contains $x + T$ whenever it contains x and, if for any $x \in D(f)$ it holds that
$$f(x + T) = f(x). \tag{1.1}$$

Remark. If also $x - T \in D(f)$ then

$$f(x - T) = f(x).$$

It follows that if T is a period of f then mT is also a period for any integer $m > 0$. The smallest value of $T > 0$ for which (1.1) holds is called the *fundamental period* of f.

For example, the functions

$$\sin \frac{m\pi x}{L}, \quad \cos \frac{m\pi x}{L}, \quad e^{i\frac{m\pi x}{L}}, \quad m = 1, 2, \ldots$$

are periodic with fundamental period $T = \frac{2L}{m}$. Note also that they are periodic with common period $2L$.

If some function f is defined on the interval $[a, a + T], T > 0$ and $f(a) = f(a + T)$, then f can be *extended periodically* with period T on the whole line as

$$f(x) := f(x - mT), \quad x \in [a + mT, a + (m + 1)T], \quad m = 0, \pm 1, \pm 2, \ldots.$$

Therefore, we may assume from now on that any periodic function is defined on the whole line.

We say that f is *p-integrable*, $1 \le p < \infty$, on the interval $[a, b]$ if

$$\int_a^b |f(x)|^p dx < \infty.$$

The set of all such functions is denoted by $L^p(a, b)$. When $p = 1$ we say that f is *integrable*. If f is p-integrable and g is p'-integrable on $[a, b]$, where

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad 1 < p < \infty, 1 < p' < \infty$$

then their product is integrable on $[a, b]$ and

$$\int_a^b |f(x)g(x)|dx \le \left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |g(x)|^{p'} dx\right)^{1/p'}.$$

This inequality is called the *Hölder's inequality* for integrals. The *Fubini's theorem* states that
$$\int_a^b \int_c^d F(x, y)dydx = \int_c^d \int_a^b F(x, y)dxdy,$$

where $F(x,y) \in L^1((a,b)\times(c,d))$ is positive.

If $f_1, f_2, \dots, f_n$ are p-integrable on $[a,b]$ for $1 \le p < \infty$ then so is their sum $\sum_{j=1}^n f_j$ and

$$\left(\int_a^b \left|\sum_{j=1}^n f_j(x)\right|^p dx\right)^{1/p} \le \sum_{j=1}^n \left(\int_a^b |f_j(x)|^p dx\right)^{1/p}.$$

This inequality is called *Minkowski's inequality*. As a consequence of Hölder's inequality we obtain the *generalized Minkowski's inequality*

$$\left(\int_a^b \left|\int_c^d F(x,y)dy\right|^p dx\right)^{1/p} \le \int_c^d \left(\int_a^b |F(x,y)|^p dx\right)^{1/p} dy. \tag{1.2}$$

Exercise 1. Prove (1.2).

Lemma 1.1. *If f is periodic with period $T > 0$ and if it is integrable on any finite interval then*

$$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx \tag{1.3}$$

for any $a \in \mathbb{R}$.

Proof. Let first $a > 0$. Then

$$\begin{aligned}\int_a^{a+T} f(x)dx &= \int_0^{a+T} f(x)dx - \int_0^a f(x)dx \\ &= \int_0^T f(x)dx + \left[\int_T^{a+T} f(x)dx - \int_0^a f(x)dx\right].\end{aligned}$$

The difference in the brackets is equal to zero due to periodicity of f. Thus, (1.3) holds for $a > 0$.

If $a < 0$ then we proceed similarly as

$$\begin{aligned}\int_a^{a+T} f(x)dx &= \int_a^0 f(x)dx + \int_0^{a+T} f(x)dx \\ &= \int_a^0 f(x)dx + \int_0^T f(x)dx - \int_{a+T}^T f(x)dx \\ &= \int_0^T f(x)dx + \left[\int_a^0 f(x)dx - \int_{a+T}^T f(x)dx\right].\end{aligned}$$

Again, the periodicity of f implies that the difference in brackets is zero. Thus lemma is proved. □

Definition 1.2. Let us assume that the domain of f is symmetric with respect to $\{0\}$, i.e. if $x \in D(f)$ then $-x \in D(f)$. A function f is called *even* if

$$f(-x) = f(x), \quad x \in D(f)$$

and *odd* if

$$f(-x) = -f(x), \quad x \in D(f).$$

Lemma 1.2. *If f is integrable on any finite interval and if it is even then*

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

for any $a > 0$. Similarly, if f is odd then

$$\int_{-a}^{a} f(x)dx = 0$$

for any $a > 0$.

Proof. Since

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{-a}^{0} f(x)dx$$

then changing variables in the second integral we obtain

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(-x)dx.$$

Now the result of this lemma follows from Definition 1.2. □

Definition 1.3. The notations $f(c \pm 0)$ are used to denote the *right and left limits*

$$f(c \pm 0) := \lim_{x \to c\pm} f(x).$$

Definition 1.4. A function f is said to be *piecewise continuous* on an interval $[a, b]$ if there are $x_0, x_1, \ldots, x_n$ such that $a = x_0 < x_1 < \cdots < x_n = b$ and

1. f is continuous on each subinterval $(x_{j-1}, x_j), j = 1, 2, \ldots, n$
2. $f(x_0 + 0), f(x_n - 0)$ and $f(x_j \pm 0), j = 1, 2, \ldots, n - 1$ exist.

Definition 1.5. A function f is said to be of *bounded variation* on an interval $[a, b]$, denoted by $BV[a, b]$, if there is $c_0 \geq 0$ such that, for any $\{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$ it holds that

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \leq c_0.$$

The number

$$V_a^b(f) := \sup_{x_0, x_1, \ldots, x_n} \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \tag{1.4}$$

is called the *full variation* of f on the interval $[a, b]$. For any $x \in [a, b]$ we can also define $V_a^x(f)$ by (1.4).

Exercise 2. Prove that

1. $V_a^x(f)$ is monotone increasing in x

2. for any $c \in (a,b)$ we have $V_a^b(f) = V_a^c(f) + V_c^b(f)$.

If f is real-valued then Exercise 2 implies that $V_a^x(f) - f(x)$ is monotone increasing in x. Indeed, for $h > 0$ we have that

$$\begin{aligned} \left(V_a^{x+h}(f) - f(x+h)\right) - \left(V_a^x(f) - f(x)\right) &= \left(V_a^{x+h}(f) - V_a^x(f)\right) - \left(f(x+h) - f(x)\right) \\ &= V_x^{x+h}(f) - \left(f(x+h) - f(x)\right) \\ &\geq V_x^{x+h}(f) - |f(x+h) - f(x)| \geq 0. \end{aligned}$$

As an immediate consequence we obtain that any real-valued function $f \in BV[a,b]$ can be represented as the difference of two monotone increasing functions as

$$f(x) = V_a^x(f) - \left(V_a^x(f) - f(x)\right).$$

This fact allows us to define the *Stieltjes integral*

$$\int_a^b g(x)df(x), \tag{1.5}$$

where $f \in BV[a,b]$ and g is an arbitrary continuous function. The integral (1.5) is defined as

$$\int_a^b g(x)df(x) = \lim_{\Delta \to 0} \sum_{j=1}^n g(\xi_j)(f(x_j) - f(x_{j-1})),$$

where $j = 1, 2, \ldots, n$, $a = x_0 < x_1 < \cdots < x_n = b$, $\xi_j \in [x_{j-1}, x_j]$ and $\Delta = \max_{1 \leq j \leq n}(x_j - x_{j-1})$.

Let us introduce the *modulus of continuity* of f by

$$\omega_h(f) := \sup_{\{x \in [a,b] : x+h \in [a,b]\}} |f(x+h) - f(x)|, \quad h > 0. \tag{1.6}$$

Definition 1.6. A function f belongs to *Hölder space* $C^\alpha[a,b], 0 < \alpha \leq 1$, if

$$\omega_h(f) \leq Ch^\alpha$$

with some constant $C > 0$. This inequality is called the *Hölder condition with exponent* α.

Definition 1.7. We say that f belongs to *Sobolev space* $W_p^1(a,b), 1 \leq p < \infty$ if $f \in L^p(a,b)$ and there is $g \in L^p(a,b)$ such that

$$f(x) = \int_a^x g(t)dt + C \tag{1.7}$$

with some constant C.

Lemma 1.3. *Suppose that $f \in W_p^1(a,b), 1 \le p < \infty$. Then f is of bounded variation. Moreover, if $p = 1$ then f is also continuous and if $1 < p < \infty$ then $f \in C^{1-1/p}[a,b]$.*

Proof. Let first $p = 1$. Then there is an integrable g such that (1.7) holds with some constant C. Hence for fixed $x \in [a,b]$ with $x + h \in [a,b]$ we have

$$f(x+h) - f(x) = \int_x^{x+h} g(t)dt.$$

It follows that

$$|f(x+h) - f(x)| = \left| \int_x^{x+h} g(t)dt \right| \to 0, \quad h \to 0$$

since g is integrable. This proves the continuity of f. At the same time for any $\{x_0, x_1, \dots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$ we have

$$\sum_{j=1}^n |f(x_j) - f(x_{j-1})| = \sum_{j=1}^n \left| \int_{x_{j-1}}^{x_j} g(t)dt \right| \le \sum_{j=1}^n \int_{x_{j-1}}^{x_j} |g(t)|dt = \int_a^b |g(t)|dt.$$

Hence, Definition 1.5 is satisfied with constant $c_0 = \int_a^b |g(t)|dt$ and f is of bounded variation.

If $1 < p < \infty$ then using Hölder's inequality for integrals we obtain for $h > 0$ that

$$\begin{aligned} |f(x+h) - f(x)| &\le \int_x^{x+h} |g(t)|dt \le \left(\int_x^{x+h} dt \right)^{1/p'} \left(\int_x^{x+h} |g(t)|^p dt \right)^{1/p} \\ &\le h^{1-1/p} \left(\int_a^b |g(t)|^p dt \right)^{1/p}, \end{aligned}$$

where $1/p + 1/p' = 1$. By Definition 1.6 it means that $f \in C^{1-1/p}[a,b]$. Lemma is proved. □

Remark. Since any $f \in W_p^1(a,b), 1 \le p < \infty$ is continuous then the constant C in (1.7) is equal to $f(a)$.

Definition 1.8. Two functions u and v are said to be *orthogonal* on $[a,b]$ if the product uv is integrable and

$$\int_a^b u(x)\overline{v(x)}dx = 0.$$

A set of functions is said to be *mutually orthogonal* if each distinct pair in the set is orthogonal on $[a,b]$.

Lemma 1.4. *The functions*

$$1, \quad \sin \frac{m\pi x}{L}, \quad \cos \frac{m\pi x}{L}, \quad m = 1, 2, \dots$$

form a mutually orthogonal set on the interval $[-L,L]$ *as well as on the interval* $[0,2L]$. *In fact,*

$$\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx=\int_{0}^{2L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx=\begin{cases}0, & m\neq n\\ L, & m=n\end{cases}, \tag{1.8}$$

$$\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx=\int_{0}^{2L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx=0, \tag{1.9}$$

$$\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx=\int_{0}^{2L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx=\begin{cases}0, & m\neq n\\ L, & m=n\end{cases} \tag{1.10}$$

and

$$\int_{-L}^{L}\sin\frac{m\pi x}{L}dx=\int_{0}^{2L}\sin\frac{m\pi x}{L}dx=\int_{-L}^{L}\cos\frac{m\pi x}{L}dx=\int_{0}^{2L}\cos\frac{m\pi x}{L}dx=0. \tag{1.11}$$

Proof. Due to Lemma 1.1 it is enough to prove the equalities (1.8), (1.9), (1.10) and (1.11) only for integrals over $[-L,L]$. Let us derive, for example, (1.9). Using the equality

$$\cos\alpha\sin\beta=\frac{1}{2}\left(\sin(\alpha+\beta)-\sin(\alpha-\beta)\right)$$

we have for $m\neq n$ that

$$\begin{aligned}\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx &= \frac{1}{2}\int_{-L}^{L}\sin\frac{(m+n)\pi x}{L}dx-\frac{1}{2}\int_{-L}^{L}\sin\frac{(m-n)\pi x}{L}dx\\ &= \frac{1}{2}\left(\frac{-\cos\frac{(m+n)\pi x}{L}}{\frac{(m+n)\pi}{L}}\right)\Bigg|_{-L}^{L}-\frac{1}{2}\left(\frac{-\cos\frac{(m-n)\pi x}{L}}{\frac{(m-n)\pi}{L}}\right)\Bigg|_{-L}^{L}=0\end{aligned}$$

since cosine is even. If $m=n$ we have

$$\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx=\frac{1}{2}\int_{-L}^{L}\sin\frac{2m\pi x}{L}dx=0$$

since sine is odd. Other identities can be proved in a similar manner and are left to the readers. Lemma is proved. □

Remark. This lemma holds also for the functions $e^{i\frac{n\pi x}{L}}, n=0,\pm1,\pm2,\ldots$ in the form

$$\int_{-L}^{L}e^{i\frac{n\pi x}{L}}e^{-i\frac{m\pi x}{L}}dx=\int_{0}^{2L}e^{i\frac{n\pi x}{L}}e^{-i\frac{m\pi x}{L}}dx=\begin{cases}0, & n\neq m\\ 2L, & n=m.\end{cases}$$

2 Formulation of Fourier series

Let us consider a series of the form

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right). \tag{2.1}$$

This series consists of $2L$-periodic functions. Thus, if the series (2.1) converges for all x, then the function to which it converges will also be $2L$-periodic. Let us denote this limiting function by $f(x)$ i.e.

$$f(x) := \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right). \tag{2.2}$$

To determine a_m and b_m we proceed as follows: assuming that the integration can be legitimately carried out term by term (it will be, for example, if $\sum_{m=1}^{\infty}(|a_m| + |b_m|) < \infty$) we obtain

$$\begin{aligned} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx &= \frac{a_0}{2} \int_{-L}^{L} \cos \frac{n\pi x}{L} dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx \\ &+ \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx \end{aligned}$$

for each fixed $n = 1, 2, \ldots$. It follows from the orthogonality relations (1.8), (1.9) and (1.11) that the only nonzero term on the right hand side is the one for which $m = n$ in the first summation. Hence

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, \ldots. \tag{2.3}$$

A similar expression for b_n is obtained by multiplying (2.2) by $\sin \frac{n\pi x}{L}$ and integrating termwise from $-L$ to L. The result is

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \ldots. \tag{2.4}$$

Using (1.11) we can easily obtain that

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx. \tag{2.5}$$

Definition 2.1. Let f be integrable (not necessarily periodic) on the interval $[-L, L]$. The *Fourier series* of f is the *trigonometric series* (2.1), where the coefficients a_0, a_m and b_m are given by (2.5), (2.3) and (2.4), respectively. In that case we write

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right). \tag{2.6}$$

Remark. This definition does not imply that the series (2.6) converges to f or that f is periodic.

Definition 2.1 and Lemma 1.2 imply that if f is even on $[-L, L]$ then the Fourier series of f has the form

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi x}{L} \tag{2.7}$$

and if f is odd then

$$f(x) \sim \sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{L}. \tag{2.8}$$

The series (2.7) and (2.8) are called the *Fourier cosine series* and *Fouries sine series*, respectively.

If $L = \pi$ then the Fourier series (2.6) ((2.7) and (2.8)) transforms to

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx), \tag{2.9}$$

where the coefficients a_0, a_m and b_m are given by (2.3), (2.4) and (2.5) with $L = \pi$.

There are different approaches if function f is defined on a nonsymmetric interval $[0, L]$ with an arbitrary $L > 0$.

1. *Even extension*. Define a function $g(x)$ on the interval $[-L, L]$ as

$$g(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L \le x < 0. \end{cases}$$

Then $g(x)$ is even and its Fourier (cosine) series (2.7) represents f on $[0, L]$.

2. *Odd extension*. Define a function $h(x)$ on the interval $[-L, L]$ as

$$h(x) = \begin{cases} f(x), & 0 \le x \le L \\ -f(-x), & -L \le x < 0. \end{cases}$$

Then $h(x)$ is odd and its Fourier (sine) series (2.8) represents f on $[0, L]$.

3. Define a function $\widetilde{f}(t)$ on the interval $[-\pi, \pi]$ as the superposition

$$\widetilde{f}(t) = f\left(\frac{tL}{2\pi} + \frac{L}{2}\right).$$

If $f(0) = f(L)$ then we may extend f to be periodic with period L. Then

$$\begin{aligned} a_0(\widetilde{f}) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \widetilde{f}(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{tL}{2\pi} + \frac{L}{2}\right) dt = \frac{1}{\pi}\frac{2\pi}{L}\int_0^L f(x)\,dx \\ &= \frac{2}{L}\int_0^L f(x)\,dx := a_0(f), \end{aligned}$$

$$a_m(\widetilde{f}) = (-1)^m \frac{2}{L}\int_0^L f(x)\cos\frac{2m\pi x}{L}dx = (-1)^m a_m(f),$$

and

$$b_m(\widetilde{f}) = (-1)^m \frac{2}{L}\int_0^L f(x)\sin\frac{2m\pi x}{L}dx = (-1)^m b_m(f).$$

Hence,

$$\widetilde{f}(t) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mt + b_m \sin mt)$$

and, at the same time,

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty}(-1)^m\left(a_m \cos\frac{2m\pi x}{L} + b_m \sin\frac{2m\pi x}{L}\right),$$

where a_0, a_m and b_m are the same and $x = \frac{tL}{2\pi} + \frac{L}{2}$.

These three alternatives allow us to consider (for simplicity) only the case of symmetric interval $[-\pi, \pi]$ such that the Fourier series will be of the form (2.9) i.e.

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx).$$

Using Euler's formula we will rewrite this series in the complex form

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{2.10}$$

where the coefficients $c_n = c_n(f)$ are equal to

$$c_n(f) = \begin{cases} \frac{a_n}{2} + \frac{b_n}{2i}, & n = 1, 2, \ldots \\ \frac{a_0}{2}, & n = 0 \\ \frac{a_{-n}}{2} - \frac{b_{-n}}{2i}, & n = -1, -2, \ldots. \end{cases} \tag{2.11}$$

The formulas (2.3), (2.4), (2.5) and (2.11) imply that

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{2.12}$$

for $n = 0, \pm 1, \pm 2, \ldots$. We call $c_n(f)$ the nth *Fourier coefficient* of f. It can be checked that

$$c_n(f) = \overline{c_{-n}(\overline{f})}. \tag{2.13}$$

Exercise 3. Prove formulas (2.10), (2.11), (2.12) and (2.13).

Exercise 4. Find the Fourier series of

a) $\operatorname{sgn}(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 0, & x = 0 \\ 1, & 0 < x \le \pi. \end{cases}$

b) $|x|, -1 \le x \le 1.$

c) $x, -1 \le x \le 1.$

d) $f(x) = \begin{cases} 0, & -L \le x \le 0 \\ L, & 0 < x \le L. \end{cases}$

Exercise 5. Suppose that

$$f(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 0, & 1 < x \le 2. \end{cases}$$

Find the Fourier cosine and sine series of $f(x)$.

Exercise 6. Show that if N is odd then $\sin^N x$ can be written as a finite sum of the form

$$\sum_{k=1}^{N} a_k \sin kx.$$

It means that this finite sum is the Fourier series of $\sin^N x$ and the coefficients a_k (which are real) are the Fourier coefficients of $\sin^N x$.

Exercise 7. Show that if N is odd then $\cos^N x$ can be written as a finite sum of the form

$$\sum_{k=1}^{N} a_k \cos kx.$$

3 Fourier coefficients and their properties

Definition 3.1. We say that a trigonometric series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

a) *converges pointwise* if for each $x \in [-\pi, \pi]$ the limit

$$\lim_{N\to\infty} \sum_{|n|\le N} c_n e^{inx}$$

exists,

b) *converges uniformly* in $x \in [-\pi, \pi]$ if the limit

$$\lim_{N\to\infty} \sum_{|n|\le N} c_n e^{inx}$$

exists uniformly,

c) *converges absolutely* if the limit

$$\lim_{N\to\infty} \sum_{|n|\le N} |c_n|$$

exists or, equivalently, if

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty.$$

These three different types of convergence appear frequently in the sequel and they are presented above from the weakest to strongest. In other words, absolute convergence implies uniform convergence which in turn implies pointwise convergence.

If f is integrable on the interval $[-\pi, \pi]$ then the Fourier coefficients $c_n(f)$ are uniformly bounded with respect to $n = 0, \pm 1, \pm 2, \dots$ i.e.

$$|c_n(f)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} f(x) e^{-inx} dx \right| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| dx, \tag{3.1}$$

where the upper bound does not depend on n. Suppose that a sequence $\{c_n\}_{n=-\infty}^{\infty}$ is such that

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty.$$

Then the series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

converges uniformly in $x \in [-\pi, \pi]$ and defines a continuous function

$$f(x) := \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{3.2}$$

whose Fourier coefficients are $\{c_n\}_{n=-\infty}^{\infty} = \{c_n(f)\}_{n=-\infty}^{\infty}$. More generally, suppose that

$$\sum_{n=-\infty}^{\infty} |n|^k |c_n| < \infty$$

for some integer $k > 0$. Then the series (3.2) defines a function which is k times differentiable with

$$f^{(k)}(x) = \sum_{n=-\infty}^{\infty} (in)^k c_n e^{inx} \tag{3.3}$$

being a continuous function. This follows from the fact that the series (3.3) converges uniformly with respect to $x \in [-\pi, \pi]$.

Let us consider one more useful example where the Fourier coefficients are applied. If $0 \le r < 1$ then the geometric progression series gives

$$\frac{1}{1 - re^{ix}} = \sum_{n=0}^{\infty} r^n e^{inx} \tag{3.4}$$

and this series converges absolutely. Using the definition of the Fourier coefficients we obtain

$$r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-inx}}{1 - re^{ix}} dx, \quad n = 0, 1, 2, \ldots$$

and

$$0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{inx}}{1 - re^{ix}} dx, \quad n = 1, 2, \ldots.$$

From the representation (3.4) we may conclude also that

$$\frac{1 - r\cos x}{1 - 2r\cos x + r^2} = \operatorname{Re}\left(\frac{1}{1 - re^{ix}}\right) = \sum_{n=0}^{\infty} r^n \cos nx = \frac{1}{2} + \frac{1}{2} \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} \tag{3.5}$$

and

$$\frac{r\sin x}{1 - 2r\cos x + r^2} = \operatorname{Im}\left(\frac{1}{1 - re^{ix}}\right) = \sum_{n=1}^{\infty} r^n \sin nx = -\frac{i}{2} \sum_{n=-\infty}^{\infty} r^{|n|} \operatorname{sgn}(n) e^{inx}. \tag{3.6}$$

Exercise 8. Verify the formulas (3.5) and (3.6).

The formulas (3.5) and (3.6) can be rewritten as

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} = \frac{1-r^2}{1-2r\cos x + r^2} := P_r(x) \tag{3.7}$$

and

$$-i \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n) r^{|n|} e^{inx} = \frac{2r\sin x}{1-2r\cos x + r^2} := Q_r(x). \tag{3.8}$$

Definition 3.2. The function $P_r(x)$ is called the *Poisson kernel* while $Q_r(x)$ is called the *conjugate Poisson kernel*.

Since the series (3.7) and (3.8) converge absolutely we have

$$r^{|n|} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x) e^{-inx} dx \quad \text{and} \quad -i\operatorname{sgn}(n) r^{|n|} = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q_r(x) e^{-inx} dx,$$

where $n = 0, \pm 1, \pm 2, \ldots$.

Exercise 9. Prove that both $P_r(x)$ and $Q_r(x)$ are solutions of the *Laplace equation*

$$u_{x_1x_1} + u_{x_2x_2} = 0$$

in the disk $x_1^2 + x_2^2 < 1$, where $x_1 + ix_2 = re^{ix}$ with $0 \le r < 1$ and $x \in [-\pi, \pi]$.

Given a sequence $a_n, n = 0, \pm 1, \pm 2, \ldots$ define

$$\Delta_n a = a_n - a_{n-1}.$$

Then for any two sequences a_n and b_n and for any integers $M < N$ the formula

$$\sum_{k=M+1}^{N} a_k \Delta_k b = a_N b_N - a_M b_M - \sum_{k=M+1}^{N} b_{k-1} \Delta_k a \tag{3.9}$$

holds. The formula (3.9) is called *summation by parts*.

Exercise 10. Prove (3.9).

Summation by parts allows us to investigate the convergence of special type of trigonometric series.

Theorem 1. *Suppose that* $c_n > 0, n = 0, 1, 2, \ldots,$ $c_n \ge c_{n+1}$ *and* $\lim_{n\to\infty} c_n = 0$. *Then the trigonometric series*

$$\sum_{n=0}^{\infty} c_n e^{inx} \tag{3.10}$$

converges for any $x \in [-\pi, \pi] \setminus \{0\}$.

Proof. Let $b_n = \sum_{k=0}^{n} e^{ikx}$. Since

$$b_n = \frac{1 - e^{ix(n+1)}}{1 - e^{ix}}, \quad x \neq 0$$

then

$$|b_n| \leq \frac{2}{|1 - e^{ix}|} = \frac{1}{|\sin \frac{x}{2}|}$$

for $x \in [-\pi, \pi] \setminus \{0\}$. Applying (3.9) shows that for $M < N$ we have

$$\begin{aligned} \sum_{k=M+1}^{N} c_k e^{ikx} &= \sum_{k=M+1}^{N} c_k \left(\sum_{l=0}^{k} e^{ilx} - \sum_{l=0}^{k-1} e^{ilx} \right) = \sum_{k=M+1}^{N} c_k \Delta_k b \\ &= c_N b_N - c_M b_M - \sum_{k=M+1}^{N} b_{k-1} \Delta_k c. \end{aligned}$$

Thus

$$\begin{aligned} \left| \sum_{k=M+1}^{N} c_k e^{ikx} \right| &\leq c_N |b_N| + c_M |b_M| + \sum_{k=M+1}^{N} |b_{k-1}| \Delta_k c \\ &\leq \frac{1}{|\sin \frac{x}{2}|} \left(c_N + c_M + \sum_{k=M+1}^{N} |c_k - c_{k-1}| \right) \\ &= \frac{1}{|\sin \frac{x}{2}|} (c_N + c_M + c_M - c_N) = \frac{2c_M}{|\sin \frac{x}{2}|} \to 0 \end{aligned}$$

as $N > M \to \infty$. This proves the theorem. □

Corollary. *Under the same assumptions as in Theorem 1, the trigonometric series (3.10) converges uniformly for all $\pi \geq |x| \geq \delta > 0$.*

Proof. If $\pi \geq |x| \geq \delta > 0$ then $\left|\sin \frac{x}{2}\right| \geq \frac{2}{\pi} \frac{|x|}{2} \geq \frac{\delta}{\pi}$. □

Theorem 1 implies that, for example, the series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{\log(2+n)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin nx}{\log(2+n)}$$

converge for all $x \in [-\pi, \pi] \setminus \{0\}$.

Modulus of continuity and tail sum

For the trigonometric series (2.10) with

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty$$

we introduce the *tail sum* by

$$E_n := \sum_{|k|>n} |c_k|, \quad n = 0, 1, 2, \dots . \tag{3.11}$$

There is a good connection between the modulus of continuity (1.6) and (3.11). Indeed, if $f(x)$ denotes the series (2.10) and $h > 0$ we have

$$\begin{aligned}
|f(x+h) - f(x)| &\le \sum_{n=-\infty}^{\infty} |c_n||e^{inh} - 1| = \sum_{|n|h\le 1} |c_n||e^{inh} - 1| + \sum_{|n|h>1} |c_n||e^{inh} - 1| \\
&\le h \sum_{|n|\le[1/h]} |n||c_n| + 2 \sum_{|n|>[1/h]} |c_n| := I_1 + I_2,
\end{aligned}$$

where $[x]$ denotes the entire part of x. If we denote $[1/h]$ by N_h then $I_2 = 2E_{N_h}$ and

$$I_1 = h \sum_{|n|\le N_h} |n| \left(E_{|n|-1} - E_{|n|}\right) = -h \sum_{l=1}^{N_h} l \left(E_l - E_{l-1}\right).$$

Using (3.9) in the latter sum we obtain

$$I_1 = -h \left(N_h E_{N_h} - 0 \cdot E_0 - \sum_{n=1}^{N_h} E_{n-1}(n - (n-1)) \right) = -h N_h E_{N_h} + h \sum_{n=1}^{N_h} E_{n-1}.$$

Since $hN_h = h[1/h] \le h \cdot \frac{1}{h} = 1$ then these formulas for I_1 and I_2 imply that

$$\omega_h(f) \le 2E_{N_h} - hN_h E_{N_h} + h \sum_{n=1}^{N_h} E_{n-1} \le 2E_{N_h} + \frac{1}{N_h} \sum_{n=0}^{N_h-1} E_n. \tag{3.12}$$

Since $E_n \to 0$ as $n \to \infty$ then the inequality (3.12) implies that $\omega_h(f) \to 0$ as $h \to 0$. Moreover, if $E_n = \mathcal{O}(n^{-\alpha})$ for some $0 < \alpha \le 1$ then

$$\omega_h(f) = \begin{cases} \mathcal{O}(h^\alpha), & 0 < \alpha < 1 \\ \mathcal{O}(h \log \frac{1}{h}), & \alpha = 1. \end{cases} \tag{3.13}$$

Here and throughout, the notation $A = \mathcal{O}(B)$ on a set X means that $|A| \le C|B|$ on the set X with some constant $C > 0$. Similarly, $A = o(B)$ means that $A/B \to 0$.

Exercise 11. Prove the second relation in (3.13).

We summarize (3.13) as follows: if the tail of the trigonometric series (2.10) behaves as $\mathcal{O}(n^{-\alpha})$ for some $0 < \alpha < 1$ then the function f to which it converges belongs to Hölder space $C^\alpha[-\pi, \pi]$.

4 Convolution and Parseval equality

Let the trigonometric series (2.10) be such that

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty.$$

Then the function f to which it converges is continuous and periodic. If $g(x)$ is any continuous function then the product fg is also continuous and hence integrable on $[-\pi, \pi]$ and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)dx = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n(f)\int_{-\pi}^{\pi} g(x)e^{inx}dx = \sum_{n=-\infty}^{\infty} c_n(f)c_{-n}(g), \tag{4.1}$$

where integration of the series term by term is justified by the uniform convergence of Fourier series. Putting $g = \overline{f}$ in (4.1) yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} c_n(f)c_{-n}(\overline{f}) = \sum_{n=-\infty}^{\infty} |c_n(f)|^2 \tag{4.2}$$

by (2.13).

Definition 4.1. Equality (4.2) is called the *Parseval equality* for the trigonometric Fourier series.

The formula (4.1) can be generalized as follows.

Exercise 12. Let periodic f be defined by absolutely convergent Fourier series (2.10) and let g be integrable and periodic. Prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(y-x)dx = \sum_{n=-\infty}^{\infty} c_n(f)c_n(g)e^{iny}$$

and that this series converges absolutely.

The generalization of Exercise 12 is given by the following theorem.

Theorem 2. *If f_1 and f_2 are two periodic L^1-functions then*

$$c_n(f_1)c_n(f_2) = c_n(f_1 * f_2),$$

where $f_1 * f_2$ denotes the convolution

$$(f_1 * f_2)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(y)f_2(x-y)dy \tag{4.3}$$

and where the integral converges for almost every x.

Proof. We note first that the convolution (4.3) is well defined as an L^1-function by the Fubini theorem. Indeed,

$$\begin{aligned}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}|f_1(y)|\cdot|f_2(x-y)|dy\right)dx &= \int_{-\pi}^{\pi}|f_1(y)|\left(\int_{-\pi-y}^{\pi-y}|f_2(z)|dz\right)dy\\ &= \int_{-\pi}^{\pi}|f_1(y)|\left(\int_{-\pi}^{\pi}|f_2(z)|dz\right)dy\end{aligned}$$

by Lemma 1.1. The Fourier coefficients of the convolution (4.3) are equal to

$$\begin{aligned}c_n(f_1*f_2) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f_1*f_2)(x)e^{-inx}dx = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}f_1(y)f_2(x-y)dy\right)e^{-inx}dx\\ &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}f_1(y)\left(\int_{-\pi}^{\pi}f_2(x-y)e^{-inx}dx\right)dy\\ &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}f_1(y)\left(\int_{-\pi-y}^{\pi-y}f_2(z)e^{-in(y+z)}dz\right)dy\\ &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}f_1(y)e^{-iny}\left(\int_{-\pi}^{\pi}f_2(z)e^{-inz}dz\right)dy = c_n(f_1)c_n(f_2)\end{aligned}$$

by Lemma 1.1. Thus, theorem is proved. □

Exercise 13. Prove that if f_1 and f_2 are integrable and periodic then their convolution is symmetric and periodic.

Exercise 14. Let f be a periodic L^1-function. Prove that

$$(f*P_r)(x) = (P_r*f)(x) = \sum_{n=-\infty}^{\infty} r^{|n|}c_n(f)e^{inx}$$

and that P_r*f satisfies the Laplace equation i.e.

$$(P_r*f)_{x_1x_1} + (P_r*f)_{x_2x_2} = 0,$$

where $x_1^2 + x_2^2 = r^2 < 1$ with $x_1 + ix_2 = re^{ix}$.

Remark. We are going to prove in Chapter 10 that for any periodic and continuous function f the limit

$$\lim_{r\to 1-}(f*P_r)(x) = f(x)$$

exists uniformly in x.

5 Fejér means of Fourier series. Uniqueness of the Fourier series.

Let us denote the *partial sum of Fourier series* of $f \in L^1(-\pi,\pi)$ by

$$S_N(f) := \sum_{|n|\le N} c_n(f)e^{inx}$$

for each $N = 0, 1, 2, \ldots$. The *Fejér means* are defined by

$$\sigma_N(f) := \frac{S_0(f) + \cdots + S_N(f)}{N+1}.$$

Writing this out in detail we see that

$$\begin{aligned}(N+1)\sigma_N(f) &= \sum_{n=0}^{N}\sum_{|k|\le n} c_k(f)e^{ikx} = \sum_{|k|\le N}\sum_{n=|k|}^{N} c_k(f)e^{ikx}\\ &= \sum_{|k|\le N}(N+1-|k|)c_k(f)e^{ikx}\end{aligned}$$

which gives the useful representation

$$\sigma_N(f) = \sum_{|k|\le N}\left(1 - \frac{|k|}{N+1}\right) c_k(f)e^{ikx}. \tag{5.1}$$

The *Fejér kernel* is

$$K_N(x) := \sum_{|k|\le N}\left(1 - \frac{|k|}{N+1}\right) e^{ikx}. \tag{5.2}$$

The sum (5.2) can be calculated precisely as

$$K_N(x) = \frac{1}{N+1}\left(\frac{\sin\frac{N+1}{2}x}{\sin\frac{x}{2}}\right)^2. \tag{5.3}$$

Exercise 15. Prove the identity (5.3).

Exercise 16. Prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)dx = 1. \tag{5.4}$$

We can rewrite $\sigma_N(f)$ from (5.1) also as

$$\sigma_N(f) = \sum_{k=-\infty}^{\infty} 1_{[-N,N]}(k)\left(1 - \frac{|k|}{N+1}\right) c_k(f)e^{ikx},$$

where

$$1_{[-N,N]}(k) = \begin{cases} 1, & |k| \le N \\ 0, & |k| > N. \end{cases}$$

Let us assume now that f is periodic. Then Exercise 12 and Theorem 2 lead to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)K_N(x-y)dy = \sum_{k=-\infty}^{\infty} 1_{[-N,N]}(k)\left(1 - \frac{|k|}{N+1}\right) c_k(f)e^{ikx}. \tag{5.5}$$

Exercise 17. Prove (5.5).

Hence, the Fejér means can be represented as

$$\sigma_N(f)(x) = (f * K_N)(x) = (K_N * f)(x). \tag{5.6}$$

The properties (5.3), (5.4) and (5.6) allow us to prove

Theorem 3. *Let $f \in L^p(-\pi,\pi)$ be periodic with $1 \le p < \infty$. Then*

$$\lim_{N\to\infty}\left(\int_{-\pi}^{\pi} |\sigma_N(f)(x) - f(x)|^p \, dx\right)^{1/p} = 0. \tag{5.7}$$

If, in addition, f has right and left limits $f(x_0 \pm 0)$ at a point $x_0 \in [-\pi,\pi]$ then

$$\lim_{N\to\infty} \sigma_N(f)(x_0) = \frac{1}{2}\left(f(x_0+0) + f(x_0-0)\right). \tag{5.8}$$

Proof. Let us first prove (5.7). Indeed, (5.4) and (5.6) give

$$\begin{aligned}
\left(\int_{-\pi}^{\pi} |\sigma_N(f)(x) - f(x)|^p \, dx\right)^{1/p} &= \left(\int_{-\pi}^{\pi} |(f * K_N)(x) - f(x)|^p \, dx\right)^{1/p} \\
&= \left(\int_{-\pi}^{\pi} \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)f(x-y)dy - \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)f(x)dy\right|^p dx\right)^{1/p} \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} \left|\int_{-\pi}^{\pi} K_N(y)(f(x-y) - f(x))dy\right|^p dx\right)^{1/p} \\
&\le \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} \left|\int_{|y|<\delta} K_N(y)(f(x-y) - f(x))dy\right|^p dx\right)^{1/p} \\
&+ \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} \left|\int_{\pi\ge|y|>\delta} K_N(y)(f(x-y) - f(x))dy\right|^p dx\right)^{1/p} := I_1 + I_2.
\end{aligned}$$

Using the generalized Minkowski's inequality we obtain that

$$\begin{aligned}
I_1 &\le \frac{1}{2\pi}\int_{|y|<\delta} K_N(y)\left(\int_{-\pi}^{\pi} |f(x-y) - f(x)|^p \, dx\right)^{1/p} dy \\
&\le \sup_{|y|<\delta}\left(\int_{-\pi}^{\pi} |f(x-y) - f(x)|^p \, dx\right)^{1/p} \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)dy \\
&= \sup_{|y|<\delta}\left(\int_{-\pi}^{\pi} |f(x-y) - f(x)|^p \, dx\right)^{1/p} \to 0
\end{aligned} \tag{5.9}$$

as $\delta \to 0$ since $f \in L^p(-\pi, \pi)$. Quite similarly,

$$
\begin{aligned}
I_2 &\le \frac{1}{2\pi} \int_{\pi \ge |y| > \delta} K_N(y) \left(\int_{-\pi}^{\pi} |f(x-y) - f(x)|^p \, dx \right)^{1/p} dy \\
&\le 2 \left(\int_{-\pi}^{\pi} |f(x)|^p \, dx \right)^{1/p} \frac{1}{2\pi} \int_{\pi \ge |y| > \delta} K_N(y) dy \qquad (5.10)
\end{aligned}
$$

since f is periodic. The next step is to note that

$$
\sin^2 \frac{y}{2} \ge \left(\frac{2}{\pi} \frac{|y|}{2} \right)^2 \ge \left(\frac{\delta}{\pi} \right)^2
$$

for $\pi \ge |y| > \delta$. That's why (5.3) leads to

$$
K_N(y) \le \frac{1}{N+1} \cdot \frac{1}{\sin^2 \frac{y}{2}} \le \frac{1}{N+1} \cdot \frac{\pi^2}{\delta^2}
$$

and

$$
\frac{1}{2\pi} \int_{\pi \ge |y| > \delta} K_N(y) dy \le \frac{\pi^2}{2\pi\delta^2} \cdot \frac{2(\pi - \delta)}{N+1} < \frac{\pi^2}{\delta^2 (N+1)} < \frac{\pi^2}{\sqrt{N}} \to 0
$$

as $N \to \infty$ if we choose $\delta = N^{-1/4}$. From (5.9) and (5.10) we may conclude that (5.7) is proved. In order to prove (5.8) we use (5.4) to represent the difference

$$
\begin{aligned}
&\sigma_N(f)(x_0) - \frac{1}{2} \left(f(x_0 + 0) + f(x_0 - 0) \right) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(y) \left[f(x_0 - y) - \frac{1}{2} \left(f(x_0 + 0) + f(x_0 - 0) \right) \right] dy \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(y) g(y) dy, \qquad (5.11)
\end{aligned}
$$

where

$$
g(y) = f(x_0 - y) - \frac{1}{2} \left(f(x_0 + 0) + f(x_0 - 0) \right).
$$

Since Fejér kernel $K_N(y)$ is even we can rewrite the right hand side of (5.11) as

$$
\frac{1}{2\pi} \int_0^{\pi} K_N(y) h(y) dy,
$$

where $h(y) = g(y) + g(-y)$. It is clear that $h(y)$ is L^1-function. But we have more, namely,

$$
\lim_{y \to 0+} h(y) = 0. \qquad (5.12)
$$

Our task now is to prove that

$$
\lim_{N \to \infty} \frac{1}{2\pi} \int_0^{\pi} K_N(y) h(y) dy = 0.
$$

We will proceed as in the proof of (5.7) i.e. we split

$$\frac{1}{2\pi}\int_0^\pi K_N(y)h(y)dy = \frac{1}{2\pi}\int_0^\delta K_N(y)h(y)dy + \frac{1}{2\pi}\int_\delta^\pi K_N(y)h(y)dy := I_1 + I_2.$$

The first term can be estimated as

$$|I_1| \le \frac{1}{2\pi}\sup_{0\le y\le\delta}|h(y)|\int_0^\pi K_N(y)dy = \frac{1}{2}\sup_{|y|\le\delta}|h(y)| \to 0$$

as $\delta \to 0+$ due to (5.12). For I_2 we have

$$|I_2| \le \frac{1}{2\pi}\left(\frac{\pi}{\delta}\right)^2\frac{1}{N+1}\int_0^\pi |h(y)|dy \to 0$$

as $N \to \infty$ if we choose, for example, $\delta = N^{-1/4}$. Thus, theorem is completely proved. □

Corollary 1. *If f is periodic and continuous on the interval $[-\pi, \pi]$ then*

$$\lim_{N\to\infty}\sigma_N(f)(x) = f(x)$$

uniformly in $x \in [-\pi, \pi]$.

Corollary 2. *Any periodic L^p-function, $1 \le p < \infty$, can be approximated by the trigonometric polynomials $\sum_{|k|\le N} b_k e^{ikx}$ (which are infinitely many times differentiable functions).*

Theorem 4 (Uniqueness of Fourier series)**.** *If periodic $f \in L^1(-\pi, \pi)$ has Fourier coefficients identically zero then $f = 0$ almost everywhere.*

Proof. Since

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}|\sigma_N(f)(x) - f(x)|\,dx = 0$$

by (5.7) then in the case when all Fourier coefficients are zero, we have

$$\int_{-\pi}^{\pi}|f(x)|\,dx = 0.$$

It means that $f = 0$ almost everywhere. This proves the theorem. □

6 Riemann-Lebesgue lemma

Theorem 5 (Riemann-Lebesgue lemma)**.** *If f is periodic with period 2π and belongs to $L^1(-\pi,\pi)$ then*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x+z)e^{-inz}dz = 0 \tag{6.1}$$

uniformly in $x\in\mathbb{R}$. In particular, $c_n(f)\to 0$ as $n\to\infty$.

Proof. Since f is periodic with period 2π then

$$\int_{-\pi}^{\pi} f(x+z)e^{-inz}dz = \int_{-\pi+x}^{\pi+x} f(y)e^{-in(y-x)}dy = e^{inx}\int_{-\pi}^{\pi} f(y)e^{-iny}dy \tag{6.2}$$

by Lemma 1.1. Formula (6.2) shows that to prove (6.1) it is enough to show that the Fourier coefficients $c_n(f)$ tend to zero as $n\to\infty$. Indeed,

$$2\pi c_n(f) = \int_{-\pi}^{\pi} f(y)e^{-iny}dy = \int_{-\pi+\pi/n}^{\pi+\pi/n} f(y)e^{-iny}dy = \int_{-\pi}^{\pi} f(t+\pi/n)e^{-int}e^{-i\pi}dt$$

by Lemma 1.1. Hence

$$-4\pi c_n(f) = \int_{-\pi}^{\pi} \left(f(t+\pi/n) - f(t)\right)e^{-int}dt. \tag{6.3}$$

If f is continuous on the interval $[-\pi,\pi]$ then

$$\sup_{t\in[-\pi,\pi]} |f(t+\pi/n) - f(t)| \to 0, \quad n\to\infty.$$

Hence $c_n(f)\to 0$ as $n\to\infty$. In case f is an arbitrary L^1-function we let $\varepsilon>0$. Then we can define a continuous function g (see Corollary 2 of Theorem 3) such that

$$\int_{-\pi}^{\pi} |f(x)-g(x)|\,dx < \varepsilon.$$

Write

$$c_n(f) = c_n(g) + c_n(f-g).$$

The first term tends to zero as $n\to\infty$ since g is continuous, whereas the second term is less than $\varepsilon/(2\pi)$. It implies that

$$\sup\lim_{n\to\infty} |c_n(f)| \le \frac{\varepsilon}{2\pi}.$$

Since ε is arbitrary then

$$\lim_{n\to\infty} |c_n(f)| = 0.$$

This fact together with (6.2) gives (6.1). Theorem is thus proved. □

Corollary. *Let f be as in Theorem 5. If periodic g is continuous on $[-\pi,\pi]$ then*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x+z)g(z)e^{-inz}dz = 0$$

and

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x+z)g(z)\sin(nz)dz = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x+z)g(z)\cos(nz)dz = 0$$

uniformly in $x \in [-\pi,\pi]$.

Exercise 18. Prove this Corollary.

Exercise 19. Show that if f satisfies the Hölder condition with exponent $\alpha \in (0,1]$ then $c_n(f) = \mathcal{O}(|n|^{-\alpha})$ as $n \to \infty$.

Exercise 20. Suppose that f satisfies the Hölder condition with exponent $\alpha > 1$. Prove that $f \equiv$ constant.

Exercise 21. Let $f(x) = |x|^\alpha$, where $-\pi \le x \le \pi$ and $0 < \alpha < 1$. Prove that $c_n(f) \asymp |n|^{-1-\alpha}$ as $n \to \infty$.

Remark. The notation $a \asymp b$ means that there exist $0 < c_1 < c_2$ such that

$$c_1|a| < |b| < c_2|a|.$$

Let us introduce for any $1 \le p < \infty$ and for any periodic $f \in L^p(-\pi,\pi)$ the *L^p-modulus of continuity* of f by

$$\omega_{p,\delta}(f) := \sup_{|h|\le\delta}\left(\int_{-\pi}^{\pi}|f(x+h)-f(x)|^p\,dx\right)^{1/p}.$$

The equality (6.3) leads to

$$\begin{aligned}|c_n(f)| &\le \frac{1}{4\pi}\int_{-\pi}^{\pi}|f(x+\pi/n)-f(x)|\,dx\\ &\le \frac{(2\pi)^{1-1/p}}{4\pi}\left(\int_{-\pi}^{\pi}|f(x+\pi/n)-f(x)|^p\,dx\right)^{1/p} \le \frac{1}{2}(2\pi)^{-1/p}\omega_{p,\pi/n}(f),\end{aligned}$$

where we have used Hölder's inequality in the penultimate step.

Exercise 22. Suppose that $\omega_{p,\delta}(f) \le C\delta^\alpha$ for some $C > 0$ and $\alpha > 1$. Prove that f is constant almost everywhere.

Hint. First show that $\omega_{p,2\delta}(f) \le 2\omega_{p,\delta}(f)$, then iterate this to obtain a contradiction.

Suppose that $f \in L^1(-\pi, \pi)$ but not necessarily periodic. We can consider Fourier series corresponding to f i.e.

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where c_n are the Fourier coefficients $c_n(f)$. The series in the right hand side is considered formally in the sense that we know nothing about its convergence. However, the limit

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \sum_{|n|\le N} c_n(f) e^{inx} dx = \int_{-\pi}^{\pi} f(x) dx \tag{6.4}$$

exists. Indeed,

$$\begin{aligned}\int_{-\pi}^{\pi} \sum_{|n|\le N} c_n(f) e^{inx} dx &= c_0(f) \int_{-\pi}^{\pi} dx + \sum_{0<|n|\le N} c_n(f) \int_{-\pi}^{\pi} e^{inx} dx = 2\pi c_0(f) \\ &= \int_{-\pi}^{\pi} f(x) dx.\end{aligned}$$

The existence of the limit (6.4) shows us that we can always integrate the Fourier series of an L^1-function term by term.

7 Fourier series of square-integrable function. Riesz-Fischer theorem.

The set of *square-integrable functions* $L^2(-\pi,\pi)$ is a linear Euclidean space equipped with inner product

$$(f,g)_{L^2(-\pi,\pi)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx. \tag{7.1}$$

We can measure the degree of approximation by the *mean square distance*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-g(x)|^2dx = (f-g,f-g)_{L^2(-\pi,\pi)}.$$

In particular, if $g(x)=\sum_{|n|\le N} b_n e^{inx}$ is a trigonometric polynomial then

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-g(x)|^2dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx + \frac{1}{2\pi}\int_{-\pi}^{\pi}|g(x)|^2dx \\
&- \frac{1}{2\pi}2\mathrm{Re}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx + \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{|n|\le N} b_n e^{inx}\sum_{|k|\le N}\overline{b_k}e^{-ikx}dx \\
&- \frac{1}{2\pi}2\mathrm{Re}\int_{-\pi}^{\pi} f(x)\sum_{|n|\le N}\overline{b_n}e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx + \frac{1}{2\pi}\sum_{|n|\le N}|b_n|^2\int_{-\pi}^{\pi}dx \\
&- 2\mathrm{Re}\sum_{|n|\le N}\overline{b_n}c_n(f) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx + \sum_{|n|\le N}|b_n|^2 - 2\mathrm{Re}\sum_{|n|\le N}\overline{b_n}c_n(f) \\
&+ \sum_{|n|\le N}|c_n(f)|^2 - \sum_{|n|\le N}|c_n(f)|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx - \sum_{|n|\le N}|c_n(f)|^2 + \sum_{|n|\le N}|b_n - c_n(f)|^2.
\end{aligned}$$

This equality has the following consequences:

1. The minimum error is

$$\min_{g(x)=\sum_{|n|\le N} b_n e^{inx}} \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-g(x)|^2dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx - \sum_{|n|\le N}|c_n(f)|^2 \tag{7.2}$$

and it is attained when $b_n = c_n(f)$.

2. For $N = 1, 2, \dots$ it is true that

$$\sum_{|n| \le N} |c_n(f)|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

and, in particular,

$$\sum_{n=-\infty}^{\infty} |c_n(f)|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx. \tag{7.3}$$

This inequality is called *Bessel's inequality*.

It turns out that (7.3) holds with equality sign. This is *Parseval equality* for $f \in L^2(-\pi, \pi)$ which we state as

Theorem 6. *For any periodic $f \in L^2(-\pi, \pi)$ with period 2π its Fourier series converges in $L^2(-\pi, \pi)$ i.e.*

$$\lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(x) - \sum_{|n| \le N} c_n(f) e^{inx} \right|^2 dx = 0 \tag{7.4}$$

and the Parseval equality

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n(f)|^2 \tag{7.5}$$

holds.

Proof. By Bessel's inequality (7.3) we have for any $f \in L^2(-\pi, \pi)$ that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{|n| \le N} c_n(f) e^{inx} - \sum_{|n| \le M} c_n(f) e^{inx} \right|^2 dx = \sum_{M+1 \le |n| \le N} |c_n(f)|^2 \to 0$$

as $N > M \to \infty$. Due to completeness of trigonometric polynomials in $L^2(-\pi, \pi)$ (see Corollary 2 of Theorem 3) we may conclude now that there is $F \in L^2(-\pi, \pi)$ such that

$$\lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| F(x) - \sum_{|n| \le N} c_n(f) e^{inx} \right|^2 dx = 0.$$

It remains to show that $F(x) = f(x)$ almost everywhere. To do this, we compute the Fourier coefficients $c_n(F)$ by writing

$$\begin{aligned} 2\pi c_n(F) &= \int_{-\pi}^{\pi} F(x) e^{-inx} dx = \int_{-\pi}^{\pi} \left(F(x) - \sum_{|k| \le N} c_k(f) e^{ikx} \right) e^{-inx} dx \\ &+ \sum_{|k| \le N} c_k(f) \int_{-\pi}^{\pi} e^{i(k-n)x} dx. \end{aligned}$$

If $N > |n|$ then the last sum is equal to $2\pi c_n(f)$. Thus, by Hölder's inequality,

$$\begin{aligned} 2\pi|c_n(F) - c_n(f)| &\le \int_{-\pi}^{\pi} \left| F(x) - \sum_{|k|\le N} c_k(f)e^{ikx} \right| dx \\ &\le \sqrt{2\pi} \left(\int_{-\pi}^{\pi} \left| F(x) - \sum_{|k|\le N} c_k(f)e^{ikx} \right|^2 dx \right)^{1/2} \to 0 \end{aligned}$$

as $N \to \infty$, i.e. $c_n(F) = c_n(f)$ for all $n = 0, \pm1, \pm2, \ldots$. Theorem 4 (uniqueness of Fourier series) implies now that $F = f$ almost everywhere. Parseval's equality follows from (7.2) if we let $N \to \infty$. □

Corollary (Riesz-Fischer theorem). *Suppose $\{b_n\}_{n=-\infty}^{\infty}$ is a sequence of complex numbers with $\sum_{n=-\infty}^{\infty} |b_n|^2 < \infty$. Then there is a unique $f \in L^2(-\pi,\pi)$ such that $b_n = c_n(f)$.*

Proof. Completely the same as the proof of Theorem 6. □

Theorem 7. *Suppose that $f \in L^2(-\pi,\pi)$ is periodic with period 2π and that its Fourier coefficients satisfy*

$$\sum_{n=-\infty}^{\infty} |n|^2 |c_n(f)|^2 < \infty. \tag{7.6}$$

Then $f \in W_2^1(-\pi,\pi)$ with Fourier series for $f'(x)$ as

$$f'(x) \sim \sum_{n=-\infty}^{\infty} inc_n(f)e^{inx}. \tag{7.7}$$

Proof. Since (7.6) holds then by Riesz-Fischer theorem there is a unique $g(x) \in L^2(-\pi,\pi)$ such that

$$g(x) \sim \sum_{n=-\infty}^{\infty} inc_n(f)e^{inx}.$$

Integrating term by term we obtain

$$\int_{-\pi}^{\pi} g(x)dx = 0.$$

Let $F(x) := \int_{-\pi}^{x} g(t)dt$. Then, for $n \ne 0$, we have

$$\begin{aligned} c_n(F) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\int_{-\pi}^{x} g(t)dt \right) e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) \left(\int_{t}^{\pi} e^{-inx}dx \right) dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) \left(\frac{e^{-in\pi}}{-in} + \frac{e^{-int}}{in} \right) dt = \frac{1}{in}\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-int}dt = \frac{1}{in}c_n(g). \end{aligned}$$

On the other hand, $c_n(g) = inc_n(f)$. Thus, by uniqueness of Fourier series, we obtain that $F(x) - f(x) =$ constant almost everywhere or $f'(x) = g(x)$ almost everywhere. It means that

$$f(x) = \int_{-\pi}^{x} g(t)dt + \text{constant},$$

where $g \in L^2(-\pi, \pi)$. Therefore, $f \in W_2^1(-\pi, \pi)$ and

$$f'(x) = g(x) \sim \sum_{n=-\infty}^{\infty} inc_n(f)e^{inx}.$$

□

Corollary. *Under the conditions of Theorem 7, it is true that*

$$\sum_{n=-\infty}^{\infty} |c_n(f)| < \infty.$$

Proof. Due to (7.6) we have

$$\sum_{n=-\infty}^{\infty} |c_n(f)| = |c_0(f)| + \sum_{n\neq 0} |c_n(f)| \le |c_0(f)| + \frac{1}{2}\sum_{n\neq 0} |n|^2|c_n(f)|^2 + \frac{1}{2}\sum_{n\neq 0}\frac{1}{|n|^2} < \infty,$$

where we have used the basic inequality $2ab \le a^2 + b^2$ for real numbers a and b. □

Using Parseval's equality (7.5) we can obtain for any periodic $f \in L^2(-\pi, \pi)$ and for any $N = 1, 2, \ldots$ that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \sum_{|n|\le N} c_n(f)e^{inx} - f(x) \right|^2 dx = \sum_{|n|>N} |c_n(f)|^2. \tag{7.8}$$

Exercise 23. Prove (7.8).

Using Parseval's equality again we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+h) - f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |c_n(f(x+h) - f(x))|^2 = \sum_{n=-\infty}^{\infty} |e^{ihn} - 1|^2|c_n(f)|^2. \tag{7.9}$$

Theorem 8. *Suppose $f \in L^2(-\pi, \pi)$ is periodic with period 2π. Then*

$$\sum_{|n|>N} |c_n(f)|^2 = \mathcal{O}(N^{-2\alpha}), \quad N = 1, 2, \ldots \tag{7.10}$$

with $0 < \alpha < 1$ if and only if

$$\sum_{n=-\infty}^{\infty} |e^{inh} - 1|^2|c_n(f)|^2 = \mathcal{O}(|h|^{2\alpha}) \tag{7.11}$$

for $|h|$ small enough.

Proof. From (7.9) we have for any integer $M > 0$ that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h)-f(x)|^2\,dx \le \sum_{|n|\le M} n^2h^2|c_n(f)|^2 + 4\sum_{|n|>M}|c_n(f)|^2 \tag{7.12}$$

if $Mh \le 1$. If (7.10) holds then the second sum is $\mathcal{O}(M^{-2\alpha})$. To estimate the first sum we use summation by parts. Writing

$$I_n := \sum_{|k|\le n}|c_k(f)|^2$$

we have

$$\sum_{1\le|n|\le M} n^2|c_n(f)|^2 = M^2I_M - 0\cdot I_0 - \sum_{n=1}^{M} I_{n-1}(n^2-(n-1)^2) = M^2I_M - \sum_{n=2}^{M}(2n-1)I_{n-1} - I_0.$$

By hypothesis,

$$I_\infty - I_n = \mathcal{O}(n^{-2\alpha}), \quad n = 1, 2, \ldots.$$

Thus,

$$\begin{aligned}
&M^2I_M - \sum_{n=2}^{M}(2n-1)\left(I_\infty + \mathcal{O}((n-1)^{-2\alpha})\right)\\
&\quad = M^2\left(I_\infty + \mathcal{O}(M^{-2\alpha})\right) - I_\infty\sum_{n=2}^{M}(2n-1) - \sum_{n=2}^{M}(2n-1)\mathcal{O}((n-1)^{-2\alpha})\\
&\quad = \mathcal{O}(M^{2-2\alpha}) - I_\infty\underbrace{\left(M^2 - \sum_{n=2}^{M}(2n-1)\right)}_{=1} - \sum_{n=2}^{M}(2n-1)\mathcal{O}((n-1)^{-2\alpha})\\
&\quad\quad = \mathcal{O}(M^{2-2\alpha}) + \mathcal{O}(M^{2-2\alpha}) = \mathcal{O}(M^{2-2\alpha}).
\end{aligned}$$

Exercise 24. Prove that

1. $\sum_{n=1}^{M}(2n-1) = M^2$
2. $\sum_{n=2}^{M}\mathcal{O}((2n-1)(n-1)^{-2\alpha}) = \mathcal{O}(M^{2-2\alpha})$ for $0 < \alpha < 1$.

Combining these two estimates we may conclude from (7.12) that there is $C > 0$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h)-f(x)|^2\,dx \le C\left(h^2M^{2-2\alpha} + M^{-2\alpha}\right).$$

Since $0 < \alpha < 1$ then choosing $M = [1/|h|]$ we obtain

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h)-f(x)|^2\,dx &\le C\left(h^2[1/|h|]^{2-2\alpha} + [1/|h|]^{-2\alpha}\right)\\
&\le C\left(h^2(1/|h|)^{2-2\alpha} + (1/|h| - \{1/|h|\})^{-2\alpha}\right)\\
&\le C\left(|h|^{2\alpha} + (1/|h|-1)^{-2\alpha}\right)\\
&= C\left(|h|^{2\alpha} + |h|^{2\alpha}/(1-|h|)^{2\alpha}\right) \le C|h|^{2\alpha}
\end{aligned}$$

if $|h| \leq 1/2$. Here $\{1/|h|\}$ denotes the fractional part of $1/|h|$ i.e. $\{1/|h|\} = 1/|h| - [1/|h|] \in [0, 1)$.

Conversely, if (7.11) holds then

$$\sum_{n=-\infty}^{\infty} (1 - \cos(nh))|c_n(f)|^2 \leq Ch^{2\alpha}$$

with some $C > 0$. Integrating this inequality with respect to h over the interval $[0, l], l > 0$ we have

$$\sum_{n=-\infty}^{\infty} |c_n(f)|^2 \int_0^l (1 - \cos(nh))dh \leq C \int_0^l h^{2\alpha} dh$$

or

$$\sum_{n=-\infty}^{\infty} |c_n(f)|^2 \left(l - \frac{\sin(nl)}{n} \right) \leq Cl^{2\alpha+1}$$

or

$$\sum_{n=-\infty}^{\infty} |c_n(f)|^2 \left(1 - \frac{\sin(nl)}{nl} \right) \leq Cl^{2\alpha}.$$

It follows that

$$Cl^{2\alpha} \geq \sum_{|n|l \geq 2} |c_n(f)|^2 \left(1 - \frac{\sin(nl)}{nl} \right) \geq \frac{1}{2} \sum_{|n|l \geq 2} |c_n(f)|^2. \tag{7.13}$$

Taking $l = 2/N$ for integer $N > 0$ in (7.13) yields

$$\sum_{|n| \geq N} |c_n(f)|^2 \leq CN^{-2\alpha}.$$

This finishes the proof. □

Remark. If $\alpha = 1$ then (7.11) implies (7.10) but not vice versa.

Exercise 25. Suppose that periodic $f \in L^2(-\pi, \pi)$ satisfies the condition

$$\int_{-\pi}^{\pi} |f(x+h) - f(x)|^2 \, dx \leq Ch^2$$

with some $C > 0$. Prove that

$$\sum_{n=-\infty}^{\infty} |n|^2 |c_n(f)|^2 < \infty$$

and, therefore, $f \in W_2^1(-\pi, \pi)$.

For any integrable function f periodic on the interval $[-\pi, \pi]$ let us introduce the mapping

$$f \mapsto \{c_n(f)\}_{n=-\infty}^{\infty},$$

where $c_n(f)$ are the Fourier coefficients of f. This mapping is a linear transformation. Formula (3.1) says that this mapping is bounded from $L^1(-\pi, \pi)$ to $l^\infty(\mathbb{Z})$. Here $\mathbb{Z}$ denotes all integer numbers and the *sequence space* $l^p(\mathbb{Z})$ consists of sequences $\{b_n\}_{n=-\infty}^{\infty}$ for which

$$\sum_{n=-\infty}^{\infty} |b_n|^p < \infty$$

if $1 \le p < \infty$ and $\sup_{n\in\mathbb{Z}} |b_n| < \infty$ if $p = \infty$.

The Parseval equality (7.5) shows that it is also bounded from $L^2(-\pi, \pi)$ to $l^2(\mathbb{Z})$. Due to an interpolation theorem we may conclude that this mapping is bounded from $L^p(-\pi, \pi)$ to $l^{p'}(\mathbb{Z})$ for any $1 < p < 2, 1/p + 1/p' = 1$ and

$$\sum_{n=-\infty}^{\infty} |c_n(f)|^{p'} \le c_p \left(\int_{-\pi}^{\pi} |f(x)|^p dx \right)^{p'/p} < \infty.$$

8 Besov and Hölder spaces

In this chapter we will consider 2π-periodic functions f which are defined via trigonometric Fourier series in $L^2(-\pi,\pi)$ as

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n(f)e^{inx}, \tag{8.1}$$

where the Fourier coefficients $c_n(f)$ satisfy the Parseval equality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n(f)|^2, \tag{8.2}$$

that is, (8.1) can be understood in the sense of $L^2(-\pi,\pi)$ as

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}\left|f(x) - \sum_{|n|\le N} c_n(f)e^{inx}\right|^2 dx = 0. \tag{8.3}$$

We will introduce new spaces of functions (that are actually subspaces of $L^2(-\pi,\pi)$) in terms of Fourier coefficients. The motivation of such approach is the following: we proved (see Theorem 7 and Exercise 25) that periodic f belongs to $W_2^1(-\pi,\pi)$ if and only if

$$\sum_{n=-\infty}^{\infty}|n|^2|c_n(f)|^2 < \infty.$$

This fact and Parseval equality justify the following definitions.

Definition 8.1. We say that a 2π-periodic function f belongs to *Sobolev space*

$$W_2^{\alpha}(-\pi,\pi)$$

for some $\alpha \ge 0$ if

$$\sum_{n=-\infty}^{\infty}|n|^{2\alpha}|c_n(f)|^2 < \infty. \tag{8.4}$$

Definition 8.2. We say that a 2π-periodic function f belongs to *Besov space*

$$B_{2,\theta}^{\alpha}(-\pi,\pi)$$

for some $\alpha \ge 0$ and some $1 \le \theta < \infty$ if

$$\sum_{j=0}^{\infty}\left(\sum_{2^j\le|n|<2^{j+1}}|n|^{2\alpha}|c_n(f)|^2\right)^{\theta/2} < \infty. \tag{8.5}$$

Definition 8.3. We say that a 2π-periodic function f belongs to *Nikol'skii space*

$$H_2^\alpha(-\pi,\pi)$$

for some $\alpha \geq 0$ if

$$\sup_{j=0,1,2,\dots} \sum_{2^j \leq |n| < 2^{j+1}} |n|^{2\alpha} |c_n(f)|^2 < \infty. \tag{8.6}$$

Definition 8.4 (See also Definition 1.6)**.**

1. We say that a 2π-periodic function f belongs to *Hölder space* $C^\alpha[-\pi,\pi]$ for some non-integer $\alpha > 0$ if f is continuous on the interval $[-\pi,\pi]$, there is a continuous derivative $f^{(k)}$ of order $k = [\alpha]$ on the interval $[-\pi,\pi]$ and for all $h \neq 0$ small enough, we have

$$\sup_{x\in[-\pi,\pi]} |f^{(k)}(x+h) - f^{(k)}(x)| \leq C|h|^{\alpha-k}, \tag{8.7}$$

   where the constant $C > 0$ does not depend on h.

2. By the space $C^k[-\pi,\pi]$ for integer $k > 0$ we mean 2π-periodic functions f that have continuous derivatives $f^{(k)}$ of order k on the interval $[-\pi,\pi]$.

Remark. We use later the following sufficient condition (see (3.13)): if there is a constant $C > 0$ such that for each $n = 1, 2, \dots$ we have

$$\sum_{|m|\geq n} |m|^k |c_m(f)| \leq Cn^{-\alpha} \tag{8.8}$$

with some integer $k \geq 0$ and some $0 < \alpha < 1$ then f belongs to Hölder space $C^{k+\alpha}[-\pi,\pi]$.

The definitions 8.1-8.3 imply the following equalities and imbeddings:

1. $B_{2,2}^\alpha(-\pi,\pi) = W_2^\alpha(-\pi,\pi), \alpha \geq 0$.
2. $W_2^0(-\pi,\pi) = L^2(-\pi,\pi)$.
3. $B_{2,1}^\alpha(-\pi,\pi) \subset B_{2,\theta}^\alpha(-\pi,\pi) \subset H_2^\alpha(-\pi,\pi), \alpha \geq 0, 1 \leq \theta < \infty$.
4. $B_{2,\theta}^0(-\pi,\pi) \subset L^2(-\pi,\pi), 1 \leq \theta \leq 2$ and $L^2(-\pi,\pi) \subset B_{2,\theta}^0(-\pi,\pi), 2 \leq \theta < \infty$.
5. $L^2(-\pi,\pi) \subset H_2^0(-\pi,\pi)$.

Exercise 26. Prove imbeddings 3, 4 and 5.

More imbeddings are formulated in the following theorems.

Theorem 9. *If $\alpha \geq 0$ then*

$$C^\alpha[-\pi,\pi] \subset W_2^\alpha(-\pi,\pi)$$

and

$$C^\alpha[-\pi,\pi] \subset H_2^\alpha(-\pi,\pi).$$

Proof. Let us prove the first claim only for integer $\alpha \geq 0$. If $\alpha = 0$ then

$$C[-\pi, \pi] \subset L^2(-\pi, \pi) = W_2^0(-\pi, \pi).$$

If $\alpha = k > 0$ is an integer then Definition 8.4 implies that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f^{(k-1)}(x+h) - f^{(k-1)}(x)|^2 dx \leq Ch^2.$$

Using Parseval equality we obtain

$$\sum_{n=-\infty}^{\infty} |n|^{2k-2} |c_n(f)|^2 |e^{inh} - 1|^2 \leq Ch^2$$

or

$$4 \sum_{n=-\infty}^{\infty} |n|^{2k-2} |c_n(f)|^2 \sin^2(nh/2)) \leq Ch^2.$$

It follows that

$$\sum_{|nh| \leq 2} |n|^{2k-2} |c_n(f)|^2 n^2 h^2 \leq Ch^2$$

or

$$\sum_{|n| \leq 2/|h|} |n|^{2k} |c_n(f)|^2 \leq C.$$

Letting $h \to 0$ yields

$$\sum_{n=-\infty}^{\infty} |n|^{2k} |c_n(f)|^2 \leq C$$

i.e. $f \in W_2^k(-\pi, \pi)$. For non-integer α one needs to use interpolation between the spaces $C^{[\alpha]}[-\pi, \pi]$ and $C^{[\alpha]+1}[-\pi, \pi]$.

Now let us consider the second claim. As above, for $f \in C^\alpha[-\pi, \pi]$, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f^{(k)}(x+h) - f^{(k)}(x)|^2 dx \leq C|h|^{2(\alpha-k)},$$

where $k = [\alpha]$ if α is not an integer and $k = \alpha - 1$ if α is an integer. By Parseval equality,

$$\sum_{n=-\infty}^{\infty} |n|^{2k} |c_n(f)|^2 |e^{inh} - 1|^2 \leq C|h|^{2(\alpha-k)}$$

or

$$2 \sum_{n=-\infty}^{\infty} |n|^{2k} |c_n(f)|^2 (1 - \cos(nh)) \leq C|h|^{2(\alpha-k)}.$$

It suffices to consider $h > 0$. If we integrate the last inequality with respect to $h > 0$ from 0 to l then

$$\sum_{n=-\infty}^{\infty} |n|^{2k} |c_n(f)|^2 \left(1 - \frac{\sin(nl)}{nl}\right) \leq Cl^{2(\alpha-k)}.$$

For $1 \le |n|l \le 2$ we obtain

$$\sum_{1\le|n|l\le 2} |n|^{2k}|c_n(f)|^2 \le Cl^{2(\alpha-k)}$$

or, equivalently,

$$\sum_{1\le|n|l\le 2} |n|^{2k}l^{2(k-\alpha)}|c_n(f)|^2 \le C,$$

where constant $C > 0$ does not depend on l. Since $2(k-\alpha) < 0$ then it follows that

$$\sum_{1/l\le|n|\le 2/l} |n|^{2\alpha}|c_n(f)|^2 \le C$$

for any $l > 0$. Choosing $l = 2^{-j}$ we obtain

$$\sum_{2^j\le|n|\le 2^{j+1}} |n|^{2\alpha}|c_n(f)|^2 \le C$$

i.e. $f \in H_2^\alpha(-\pi,\pi)$. This finishes the proof. □

Theorem 10. *Assume that $\alpha > 1/2$ and that $\alpha - 1/2$ is not an integer. Then*

$$H_2^\alpha(-\pi,\pi) \subset C^{\alpha-1/2}[-\pi,\pi]. \tag{8.9}$$

Proof. Let $k = [\alpha]$ so that $\alpha = k + \{\alpha\}$, where $\{\alpha\}$ denotes the fractional part of α. Note that, in general, $0 \le \{\alpha\} < 1$ and in this theorem $\{\alpha\} \ne 1/2$. We will assume first that $\{\alpha\} = 0$, i.e. $\alpha = k$ is integer and $k \ge 1$. If $f \in H_2^k(-\pi,\pi)$ then there is a constant $C > 0$ such that

$$\sum_{2^j\le|m|<2^{j+1}} |m|^{2k}|c_m(f)|^2 \le C \tag{8.10}$$

for each $j = 0, 1, 2, \ldots$. Let us estimate the tail (8.8). Indeed, by Cauchy-Schwarz-Buniakowsky inequality and (8.10),

$$\begin{aligned}
\sum_{|m|\ge n} |m|^{k-1}|c_m(f)| &\le \sum_{\substack{j=j_0\\ 2^{j_0}\sim n}}^{\infty} \sum_{2^j\le|m|<2^{j+1}} |m|^{k-1}|c_m(f)| \\
&\le \sum_{\substack{j=j_0\\ 2^{j_0}\sim n}}^{\infty} \left(\sum_{2^j\le|m|<2^{j+1}} |m|^{2k}|c_m(f)|^2\right)^{1/2} \left(\sum_{2^j\le|m|<2^{j+1}} \frac{1}{m^2}\right)^{1/2} \\
&\le \sqrt{C} \sum_{\substack{j=j_0\\ 2^{j_0}\sim n}}^{\infty} \left(\sum_{2^j\le|m|<2^{j+1}} \frac{1}{m^2}\right)^{1/2} \le \sqrt{C} \sum_{\substack{j=j_0\\ 2^{j_0}\sim n}}^{\infty} 2^{-j/2} \le C2^{-j_0/2}.
\end{aligned}$$

Here and throughout, $2^{j_0} \sim n$ means that $2^{j_0} \le n < 2^{j_0+1}$. Therefore, we obtain

$$\sum_{|m|\ge n} |m|^{k-1}|c_m(f)| \le Cn^{-1/2},$$

where the constant $C > 0$ is independent of n. It means that (see (8.8)) f belongs to $C^{k-1+1/2}[-\pi,\pi] = C^{k-1/2}[-\pi,\pi]$.

If $\alpha > 1/2$ is not an integer and $\alpha - 1/2$ is not an integer then for $f \in H_2^\alpha(-\pi,\pi)$ we have instead of (8.10) the estimate

$$\sum_{2^j \le |m| < 2^{j+1}} |m|^{2k+2\{\alpha\}}|c_m(f)|^2 \le C, \tag{8.11}$$

where $k = [\alpha], 0 < \{\alpha\} < 1$ and $\{\alpha\} \ne 1/2$. If $k = 0$ then $1/2 < \alpha = \{\alpha\} < 1$. Repeating now the above procedure we obtain easily

$$\begin{aligned}\sum_{|m|\ge n} |c_m(f)| &\le \sum_{\substack{j=j_0\\2^{j_0}\sim n}}^{\infty} \left(\sum_{2^j\le|m|<2^{j+1}} |m|^{2\alpha}|c_m(f)|^2\right)^{1/2} \left(\sum_{2^j\le|m|<2^{j+1}} |m|^{-2\alpha}\right)^{1/2}\\ &\le C\sum_{\substack{j=j_0\\2^{j_0}\sim n}}^{\infty} \left(\sum_{2^j\le|m|<2^{j+1}} |m|^{-2\alpha}\right)^{1/2} \le C\sum_{\substack{j=j_0\\2^{j_0}\sim n}}^{\infty} 2^{-(\alpha-1/2)j} \le Cn^{-(\alpha-1/2)}\end{aligned}$$

i.e. we have again that $f \in C^{\alpha-1/2}[-\pi,\pi]$.

For the case $[\alpha] = k \ge 1$, α is not an integer and $\alpha - 1/2$ is not an integer either we consider two cases: $0 < \{\alpha\} < 1/2$ and $1/2 < \{\alpha\} < 1$. In the first case we have

$$\begin{aligned}\sum_{|m|\ge n} |m|^{k-1}|c_m(f)| \le \sum_{\substack{j=j_0\\2^{j_0}\sim n}}^{\infty} \left(\sum_{2^j\le|m|<2^{j+1}} |m|^{2k+2\{\alpha\}}|c_m(f)|^2\right)^{1/2} \left(\sum_{2^j\le|m|<2^{j+1}} |m|^{-2-2\{\alpha\}}\right)^{1/2}\\ \le C\sum_{\substack{j=j_0\\2^{j_0}\sim n}}^{\infty} 2^{-j-j\{\alpha\}}2^{j/2} \le Cn^{-1/2-\{\alpha\}}.\end{aligned}$$

This means again that $f \in C^{k-1+1/2+\{\alpha\}}[-\pi,\pi] = C^{\alpha-1/2}[-\pi,\pi]$. In the second case $1/2 < \{\alpha\} < 1$ we proceed as follows:

$$\begin{aligned}\sum_{|m|\ge n} |m|^{k}|c_m(f)| \le \sum_{\substack{j=j_0\\2^{j_0}\sim n}}^{\infty} \left(\sum_{2^j\le|m|<2^{j+1}} |m|^{2k+2\{\alpha\}}|c_m(f)|^2\right)^{1/2} \left(\sum_{2^j\le|m|<2^{j+1}} |m|^{-2\{\alpha\}}\right)^{1/2}\\ \le C\sum_{\substack{j=j_0\\2^{j_0}\sim n}}^{\infty} 2^{-(\{\alpha\}-1/2)j} \le Cn^{-\{\alpha\}+1/2}.\end{aligned}$$

It means that $f \in C^{k+\{\alpha\}-1/2}[-\pi,\pi] = C^{\alpha-1/2}[-\pi,\pi]$. Hence, this theorem is completely proved. □

Corollary 1. *Assume that $\alpha = k + 1/2$ for some integer $k \geq 1$. Then*

$$H_2^\alpha(-\pi,\pi) \subset C^{\beta-1/2}[-\pi,\pi] \tag{8.12}$$

for any $1/2 < \beta < \alpha$.

Corollary 2. *Assume that $\alpha > 1/2$ and $\alpha - 1/2$ is not an integer. Then*

$$B_{2,\theta}^\alpha(-\pi,\pi) \subset C^{\alpha-1/2}[-\pi,\pi]$$

for any $1 \leq \theta < \infty$.

Exercise 27. Prove Corollaries 1 and 2.

Exercise 28. Prove that the Fourier series (8.1) with coefficients

$$c_n(f) = \frac{1}{|n|\log(1+|n|)}, n \neq 0, \quad c_0(f) = 1$$

defines a function from Besov space $B_{2,\theta}^{1/2}(-\pi,\pi)$ for any $1 < \theta < \infty$ but not for $\theta = 1$.

Exercise 29. Prove that the Fourier series (8.1) with coefficients

$$c_n(f) = \frac{1}{|n|^{3/2}\log(1+|n|)}, n \neq 0, \quad c_0(f) = 1$$

defines a function from Besov space $B_{2,\theta}^{1}(-\pi,\pi)$ for any $1 < \theta < \infty$ but not for $\theta = 1$.

Exercise 30. Consider the Fourier series (8.1) with coefficients

$$c_n(f) = \frac{1}{|n|^2\log^\beta(1+|n|)}, n \neq 0, \quad c_0(f) = 1.$$

Prove that

1. $f \in H_2^{3/2}(-\pi,\pi)$ if $\beta \geq 0$
2. $f \in W_2^{3/2}(-\pi,\pi)$ if $\beta > 1/2$
3. $f \in C^1[-\pi,\pi]$ if $\beta > 1$ but $f \notin C^1[-\pi,\pi]$ if $\beta \leq 1$.

Exercise 31. Let

$$\sum_{k=0}^{\infty} a^k \cos(b^k x)$$

be a trigonometric series, where $b = 2, 3, \ldots$ and $0 < a < 1$. Prove that the series defines a function from $C^1[-\pi,\pi]$ if $0 < ab < 1$ and a function from Hölder space $C^\gamma[-\pi,\pi], \gamma < 1$ if $ab = 1$.

Exercise 32. Assume that $a = 1/b^2$ in Exercise 31. Is it true that $f \in C^1[-\pi,\pi]$?

9 Absolute convergence. Bernstein and Peetre theorems.

We prove first that if 2π-periodic function f belongs to Nikol'skii space $H_2^\alpha(-\pi,\pi)$ for some $0<\alpha<1$ in the sense of Definition 8.3 then it is equivalent to the L^2-*Hölder condition* of order α, i.e.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h)-f(x)|^2dx \le K|h|^{2\alpha} \tag{9.1}$$

with some constant $K>0$ and for all $h\neq 0$ small enough. Indeed, due to Parseval equality we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h)-f(x)|^2dx &= \sum_{n=-\infty}^{\infty}|c_n(f)|^2|e^{inh}-1|^2\\ &\le \sum_{|n|\le 2^{j_0}}|n|^2|h|^2|c_n(f)|^2+4\sum_{|n|\ge 2^{j_0}}|c_n(f)|^2, \end{aligned}\tag{9.2}$$

where j_0 is chosen so that $2^{j_0}\le\frac{1}{|h|}\le 2^{j_0+1}$. The first sum on the right-hand side of (9.2) is estimated from above as

$$\begin{aligned}|h|^2\sum_{|n|\le 2^{j_0}}|n|^2|c_n(f)|^2 &\le |h|^2\sum_{j=0}^{j_0}\sum_{2^j\le|n|<2^{j+1}}|n|^2|c_n(f)|^2\\ &\le |h|^2\sum_{j=0}^{j_0}\left(2^{j+1}\right)^{2-2\alpha}\sum_{2^j\le|n|<2^{j+1}}|n|^{2\alpha}|c_n(f)|^2\\ &\le C|h|^2\sum_{j=0}^{j_0}\left(2^{2-2\alpha}\right)^{j+1}=C|h|^2\frac{\left(2^{2-2\alpha}\right)^{j_0+2}-2^{2-2\alpha}}{2^{2-2\alpha}-1}\\ &\le C|h|^2\left(2^{j_0}\right)^{2-2\alpha}\le C|h|^2\left(\frac{1}{|h|}\right)^{2-2\alpha}=C|h|^{2\alpha}\end{aligned}\tag{9.3}$$

if $0<\alpha<1$. We have used this condition for α since we considered geometric progression with multiplier $2^{2-2\alpha}\neq 1$. The second sum on the right-hand side of (9.2) is estimated from above as

$$\begin{aligned}4\sum_{j=j_0}^{\infty}\sum_{2^j\le|n|<2^{j+1}}|n|^{2\alpha}|n|^{-2\alpha}|c_n(f)|^2 &\le 4\sum_{j=j_0}^{\infty}2^{-2\alpha j}\sum_{2^j\le|n|<2^{j+1}}|n|^{2\alpha}|c_n(f)|^2\\ &\le C\sum_{j=j_0}^{\infty}2^{-2\alpha j}\le C2^{-2j_0\alpha}\le C(2|h|)^{2\alpha}\le C|h|^{2\alpha}\end{aligned}$$

since $\frac{1}{|h|}\le 2^{j_0+1}$ and the criterion of Definition 8.3 holds. Thus, (9.1) is proved. Conversely, if the L^2-Hölder condition (9.1) is fulfilled then Theorem 8 implies for each

$N = 1, 2, \ldots$ the inequality

$$\sum_{|n|\geq N} |c_n(f)|^2 \leq CN^{-2\alpha}$$

with the same α as in (9.1). But this leads to the inequality

$$N^{2\alpha} \sum_{N\leq|n|<2N} |c_n(f)|^2 \leq C,$$

where the constant C is independent of N. Thus, we obtain for any integer $N > 0$ that

$$\sum_{N\leq|n|<2N} |n|^{2\alpha}|c_n(f)|^2 \leq C.$$

Since N is arbitrary we may conclude that $f \in H_2^\alpha(-\pi, \pi)$ for $0 < \alpha \leq 1$ in the sense of Definition 8.3. Therefore, the L^2-Hölder condition (9.1) can be considered as an equivalent definition of Nikol'skii space $H_2^\alpha(-\pi, \pi)$ for $0 < \alpha < 1$.

Exercise 33. Prove that if f belongs to Nikol'skii space $H_2^\alpha(-\pi, \pi)$ for any non-integer $\alpha > 0$ in the sense of Definition 8.3 then the following L^2-Hölder condition holds:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f^{(k)}(x+h) - f^{(k)}(x)|^2 dx \leq K|h|^{2\alpha-2k}$$

with some constant $K > 0$ and $k = [\alpha]$.

Theorem 11 (Bernstein, 1914)**.** *Assume that 2π-periodic function f satisfies the L^2-Hölder condition with $1/2 < \alpha \leq 1$. Then its trigonometric Fourier series converges absolutely, i.e.*

$$\sum_{n=-\infty}^{\infty} |c_n(f)| < \infty.$$

Proof. Since the L^2-Hölder condition (9.1) is equivalent to $f \in H_2^\alpha(-\pi, \pi)$ for $0 < \alpha < 1$ then there is a constant $C > 0$ such that

$$\sum_{2^j\leq|n|<2^{j+1}} |n|^{2\alpha}|c_n(f)|^2 \leq C \tag{9.4}$$

for each $j = 0, 1, 2, \ldots$. Hence we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |c_n(f)| &= |c_0(f)| + \sum_{j=0}^{\infty} \sum_{2^j\leq|n|<2^{j+1}} |n|^\alpha|c_n(f)||n|^{-\alpha} \\
&\leq |c_0(f)| + \sum_{j=0}^{\infty} \left(\sum_{2^j\leq|n|<2^{j+1}} |n|^{2\alpha}|c_n(f)|^2\right)^{1/2} \left(\sum_{2^j\leq|n|<2^{j+1}} |n|^{-2\alpha}\right)^{1/2} \\
&\leq |c_0(f)| + \sqrt{C}\sum_{j=0}^{\infty} 2^{-\alpha j}2^{j/2} = |c_0(f)| + \sqrt{C}\sum_{j=0}^{\infty} \left(2^{-(\alpha-1/2)}\right)^j < \infty
\end{aligned}$$

since $\alpha > 1/2$. Thus, theorem is proved. □

Corollary. *Theorem 11 holds for spaces* $C^\alpha[-\pi,\pi]$, $B^\alpha_{2,\theta}(-\pi,\pi)$ *and* $H^\alpha_2(-\pi,\pi)$ *for any* $\alpha > 1/2$ *and* $1 \le \theta < \infty$.

Exercise 34. Prove this Corollary.

Theorem 12 (Peetre, 1967)**.** *Assume that* 2π*-periodic function* f *belongs to Besov space* $B^{1/2}_{2,1}(-\pi,\pi)$. *Then its trigonometric Fourier series converges absolutely.*

Proof. If $f \in B^{1/2}_{2,1}(-\pi,\pi)$ then

$$\sum_{j=0}^{\infty}\left(\sum_{2^j\le|n|<2^{j+1}}|n||c_n(f)|^2\right)^{1/2} < \infty. \tag{9.5}$$

Hence we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty}|c_n(f)| &= |c_0(f)| + \sum_{j=0}^{\infty}\sum_{2^j\le|n|<2^{j+1}}|n|^{1/2}|c_n(f)||n|^{-1/2}\\ &\le |c_0(f)| + \sum_{j=0}^{\infty}\left(\sum_{2^j\le|n|<2^{j+1}}|n||c_n(f)|^2\right)^{1/2}\left(\sum_{2^j\le|n|<2^{j+1}}|n|^{-1}\right)^{1/2}\\ &\le |c_0(f)| + \sum_{j=0}^{\infty}\left(\sum_{2^j\le|n|<2^{j+1}}|n||c_n(f)|^2\right)^{1/2}\left(2^{-j}2^j\right)^{1/2}\\ &= |c_0(f)| + \sum_{j=0}^{\infty}\left(\sum_{2^j\le|n|<2^{j+1}}|n||c_n(f)|^2\right)^{1/2} < \infty\end{aligned}$$

due to (9.5). This proves the theorem. □

Corollary. *It is true that*

$$B^{1/2}_{2,1}(-\pi,\pi) \subset C[-\pi,\pi].$$

Exercise 35. Prove that the imbedding

$$B^{1/2}_{2,\theta}(-\pi,\pi) \subset C[-\pi,\pi]$$

does not hold for $1 < \theta < \infty$ by considering the function from Exercise 28.

Exercise 36. Prove that

$$H^\alpha_2(-\pi,\pi) \subset B^{1/2}_{2,1}(-\pi,\pi)$$

if $\alpha > 1/2$.

Theorem 13. *Assume that* 2π*-periodic function* f *belongs to Sobolev space* $W^1_p(-\pi,\pi)$ *with some* $1 < p < \infty$. *Then its trigonometric Fourier series converges absolutely.*

Proof. Since

$$W^1_{p_1}(-\pi,\pi) \subset W^1_{p_2}(-\pi,\pi)$$

for $1 \le p_2 < p_1$, then we may assume without loss of generality that $f \in W^1_p(-\pi,\pi)$ with $1 < p \le 2$. Then there is a function $g \in L^p(-\pi,\pi)$ with $1 < p \le 2$ such that

$$f(x) = \int_{-\pi}^{x} g(t)dt + f(-\pi), \quad \int_{-\pi}^{\pi} g(t)dt = 0. \tag{9.6}$$

As we know from the proof of Theorem 7, (9.6) leads to

$$c_n(f) = \frac{1}{in} c_n(g), \quad n \ne 0.$$

Since $g \in L^p(-\pi,\pi)$ with $1 < p \le 2$ then the results of Chapter 7 give

$$\left(\sum_{n=-\infty}^{\infty} |c_n(g)|^{p'}\right)^{1/p'} < \infty, \tag{9.7}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. The facts (9.6), (9.7) and Hölder's inequality imply that

$$\sum_{n=-\infty}^{\infty} |c_n(f)| = |c_0(f)| + \sum_{n\ne 0} \frac{1}{|n|} |c_n(g)| \le |c_0(f)| + \left(\sum_{n\ne 0} \frac{1}{|n|^p}\right)^{1/p} \left(\sum_{n\ne 0} |c_n(g)|^{p'}\right)^{1/p'} < \infty$$

since $1 < p \le 2$. This finishes the proof. □

Remark. For Sobolev space $W^1_1(-\pi,\pi)$ this theorem is not valid, i.e. there is a function f from $W^1_1(-\pi,\pi)$ with absolutely divergent trigonometric Fourier series. More precisely, we will prove in the next chapter that the function

$$f(x) := \sum_{n=1}^{\infty} \frac{\sin nx}{n \log(1+n)} \tag{9.8}$$

belongs to Sobolev space $W^1_1(-\pi,\pi)$, is continuous on the interval $[-\pi,\pi]$ but its trigonometric Fourier series (9.8) diverges absolutely.

The next theorem is due to Zigmund (1958–1959).

Theorem 14. *Suppose that $f \in W^1_1(-\pi,\pi) \cap C^\alpha[-\pi,\pi]$ with some $0 < \alpha < 1$. Then its trigonometric Fourier series converges absolutely.*

Proof. Since $f \in W^1_1(-\pi,\pi)$ then f is of bounded variation. The periodicity of f implies that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x+h) - f(x)|^2 dx = \frac{1}{2\pi N} \int_{-\pi}^{\pi} \sum_{k=1}^{N} |f(x+kh) - f(x+(k-1)h)|^2 dx, \tag{9.9}$$

where integer N is chosen so that $N|h| \le 1$ for $h \ne 0$ small enough. We choose $N = [1/|h|]$. Since $f \in C^\alpha[-\pi,\pi]$ the right-hand side of (9.9) can be estimted as

$$\begin{aligned}
&\frac{1}{2\pi N}\int_{-\pi}^{\pi}\sum_{k=1}^{N}|f(x+kh)-f(x+(k-1)h)|^2dx\\
&\qquad\le \frac{C|h|^\alpha}{2\pi N}\int_{-\pi}^{\pi}\sum_{k=1}^{N}|f(x+kh)-f(x+(k-1)h)|dx\\
&\qquad\le \frac{C|h|^\alpha}{2\pi N}\left(V_{-\pi-1}^{-\pi}(f)+V_{-\pi}^{\pi}(f)+V_{\pi}^{\pi+1}(f)\right)2\pi \le \frac{C|h|^\alpha}{N}\\
&\qquad= \frac{C|h|^\alpha}{[1/|h|]} = \frac{C|h|^\alpha}{1/|h|-\{1/|h|\}} \le \frac{C|h|^\alpha}{1/|h|-1} = \frac{C|h|^{\alpha+1}}{1-|h|} \le C|h|^{\alpha+1}
\end{aligned}$$

if $|h| \le 1/2$. This inequality means (see (9.9)) that $f \in H_2^{\frac{\alpha+1}{2}}(-\pi,\pi)$ with $\frac{\alpha+1}{2} > 1/2$ for $\alpha > 0$. Using Bernstein's theorem we may conclude that this theorem is proved. □

Exercise 37. Let (periodic) f be defined by

$$f(x) := \sum_{k=1}^{\infty}\frac{e^{ikx}}{k}.$$

Prove that f belongs to Nikol'skii space $H_2^{1/2}(-\pi,\pi)$ but its trigonometric Fourier series is not absolutely convergent.

Exercise 38. Let (periodic) f be defined by the absolutely convergent Fourier series

$$f(x) := \sum_{k=1}^{\infty}\frac{e^{ikx}}{k^{3/2}}.$$

1. Show that f belongs to Nikol'skii space $H_2^1(-\pi,\pi)$ but

   $$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x+h)-f(x)|^2dx \ge \frac{4h^2}{\pi^2}\log\frac{\pi}{|h|}$$

   for $0 < |h| < 1$, that is, (9.1) does not hold for $\alpha = 1$.

2. Show that f does not belong to Besov space $B_{2,\theta}^1(-\pi,\pi)$ for any $1 \le \theta < \infty$.

10 Dirichlet kernel. Pointwise and uniform convergence.

The material of this chapter makes a central part of the theory of trigonometric Fourier series. In this chapter we will give an answer to the question: to which value the trigonometric Fourier series converges?

The *Dirichlet kernel* $D_N(x)$ which is defined by symmetric finite trigonometric sum

$$D_N(x) := \sum_{|n|\le N} e^{inx} \tag{10.1}$$

plays the key role in this chapter. If $x \in [-\pi, \pi] \setminus \{0\}$ then $D_N(x)$ from (10.1) can be recalculated as follows. Using Euler's formula we have

$$\begin{aligned} D_N(x) &= \sum_{n=-N}^{N} e^{inx} = e^{-iNx} \sum_{n=-N}^{N} e^{i(n+N)x} = e^{-iNx} \sum_{k=0}^{2N} e^{ikx} = e^{-iNx} \frac{1 - e^{i(2N+1)x}}{1 - e^{ix}} \\ &= \frac{e^{-iNx} - e^{i(N+1)x}}{1 - e^{ix}} = \frac{e^{i(N+1/2)x} - e^{-i(N+1/2)x}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin(N+1/2)x}{\sin x/2}. \end{aligned}$$

Thus, the Dirichlet kernel equals

$$D_N(x) = \frac{\sin(N+1/2)x}{\sin x/2}, \quad x \ne 0. \tag{10.2}$$

For $x = 0$ we have

$$D_N(0) = 2N + 1 = \lim_{x\to 0} D_N(x),$$

so that (10.2) holds for all $x \in [-\pi, \pi]$.

Exercise 39. Prove that

1. $\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} D_N(x)dx = 1, N = 0, 1, 2, \ldots$

2. $K_N(x) = \frac{1}{N+1} \sum_{j=0}^{N} D_j(x)$, where $K_N(x)$ is the Fejér kernel (5.2).

Recall that the trigonometric Fourier partial sum is given by

$$S_N f(x) = \sum_{|n|\le N} c_n(f) e^{inx}. \tag{10.3}$$

The Fourier coefficients of $D_N(x)$ are equal to

$$c_n(D_N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} \sum_{|k|\le N} e^{ikx} dx = \begin{cases} 0, & |n| > N \\ 1, & |n| \le N. \end{cases}$$

Hence, if f is periodic and integrable then the partial sum (10.3) can be rewritten as (see Exercise 12)

$$S_N f(x) = \sum_{n=-\infty}^{\infty} c_n(D_N)c_n(f)e^{inx} = (f * D_N)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x-y)f(y)dy$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y)f(x+y)dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+y)\frac{\sin(N+1/2)y}{\sin y/2}dy. \quad (10.4)$$

Exercise 40. Let f be the function

$$f(x) = \begin{cases} \frac{1}{2} - \frac{x}{2\pi}, & 0 < x \leq \pi \\ -\frac{1}{2} - \frac{x}{2\pi}, & -\pi \leq x < 0. \end{cases}$$

Show that

1. $(S_N f)'(x) = \frac{1}{2\pi}(D_N(x) - 1)$
2. $\lim_{N\to\infty} S_N f(x) = f(x), x \neq 0$
3. $\lim_{N\to\infty} S_N f(0) = 0$.

Exercise 41. Prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)|dx = \frac{4\log N}{\pi^2} + \mathcal{O}(1).$$

Since the Dirichlet kernel is an even function (see (10.2)) we can rewrite (10.4) as

$$S_N f(x) = \frac{1}{2\pi}\int_0^{\pi} (f(x+y) + f(x-y))\frac{\sin(N+1/2)y}{\sin y/2}dy. \quad (10.5)$$

Using the normalization of the Dirichlet kernel (see Exercise 39) we have for any function $S(x)$ that

$$S_N f(x) - S(x) = \frac{1}{2\pi}\int_0^{\pi} (f(x+y) + f(x-y) - 2S(x))\frac{\sin(N+1/2)y}{\sin y/2}dy. \quad (10.6)$$

Our task is to define $S(x)$ so that the limit of the right-hand side of (10.6) is equal to zero. We will simplify the problem into two steps. The first simplification is connected with

Lemma 10.1. *For all $z \in [-\pi, \pi]$ it is true that*

$$\left|\frac{1}{\sin z/2} - \frac{2}{z}\right| \leq \frac{\pi^2}{24}. \quad (10.7)$$

Proof. First we show that

$$\left|\sin z/2 - \frac{z}{2}\right| \le \frac{|z|^3}{48} \tag{10.8}$$

for all $z \in [-\pi, \pi]$. In order to prove this inequality it is enough to show that

$$0 < x - \sin x < \frac{x^3}{6}$$

for all $0 < x < \pi/2$. The left inequality is well-known. To prove the right inequality we introduce $h(x)$ as

$$h(x) = x - \sin x - x^3/6.$$

Then its derivative satisfies

$$h'(x) = 1 - \cos x - x^2/2 = 2(\sin^2 x/2 - x^2/4) < 0$$

for all $0 < x < \pi/2$. Thus, $h(x)$ is monotone decreasing on the interval $[0, \pi/2]$. It implies that

$$0 = h(0) > h(x) = x - \sin x - x^3/6$$

for all $0 < x < \pi/2$. This proves (10.8) which in turn yields

$$\left|\frac{1}{\sin z/2} - \frac{2}{z}\right| = \frac{2\,|z/2 - \sin(z/2)|}{|z||\sin(z/2)|} \le \frac{|z|^3/24}{|z||\sin(z/2)|} \le \frac{|z|^3/24}{|z||z|/\pi} \le \frac{\pi|z|}{24} \le \frac{\pi^2}{24}$$

since $|\sin z/2| \ge |z|/\pi$ for all $z \in [-\pi, \pi]$. This finishes the proof. □

As an immediate corollary of Lemma 10.1 we obtain for any periodic and integrable function f that the function

$$y \mapsto (f(x+y) + f(x-y) - 2S(x))\left(\frac{1}{\sin y/2} - \frac{2}{y}\right)$$

is integrable on the interval $[0, \pi]$ uniformly in $x \in [-\pi, \pi]$ if $S(x)$ is bounded, i.e.

$$\begin{aligned}\int_0^\pi |f(x+y) + f(x-y) - 2S(x)| &\left|\frac{1}{\sin y/2} - \frac{2}{y}\right| dy \\ &\le \frac{\pi^2}{24}\left(\int_0^\pi |f(x+y)|dy + \int_0^\pi |f(x-y)|dy + 2\pi|S(x)|\right) \\ &\le \frac{\pi^2}{24}\left(\int_{-\pi}^\pi |f(y)|dy + 2\pi \sup_{x\in[-\pi,\pi]} |S(x)|\right).\end{aligned}$$

Application of Riemann-Lebesgue lemma (Theorem 5) gives us that

$$\lim_{N\to\infty} \frac{1}{2\pi}\int_0^\pi (f(x+y) + f(x-y) - 2S(x))\left(\frac{1}{\sin y/2} - \frac{2}{y}\right)\sin(N+1/2)y = 0 \tag{10.9}$$

pointwise in $x \in [-\pi, \pi]$ and even uniformly in $x \in [-\pi, \pi]$ if $S(x)$ is bounded on the interval $[-\pi, \pi]$.

Thus, we have reduced the question of pointwise or uniform convergence in (10.6) to proving that

$$\lim_{N\to\infty} \int_0^\pi (f(x+y) + f(x-y) - 2S(x)) \frac{\sin(N+1/2)y}{y} dy = 0 \tag{10.10}$$

pointwise or uniformly in $x \in [-\pi, \pi]$.

For the second simplification we consider the contribution to (10.10) from the interval $0 < \delta \le y \le \pi$. Note that the function

$$\frac{f(x+y) + f(x-y) - 2S(x)}{y}$$

is integrable in y on the interval $[\delta, \pi]$ uniformly in $x \in [-\pi, \pi]$ if $S(x)$ is bounded. Hence, by the Riemann-Lebesgue lemma this contribution tends to zero when $N \to \infty$. We summarize these two simplifications as

$$S_N f(x) - S(x) = \frac{1}{\pi} \int_0^\delta (f(x+y) + f(x-y) - 2S(x)) \frac{\sin(N+1/2)y}{y} dy + o_N(1) \tag{10.11}$$

pointwise or uniformly in $x \in [-\pi, \pi]$.

Let us assume now that f is a piecewise continuous periodic function. The question is: to which values of $S(x)$ in (10.11) the trigonometric Fourier partial sum may converge? The second part of Theorem 3 shows that the Fejér means converge to

$$\lim_{N\to\infty} \sigma_N f(x) = \frac{1}{2} (f(x+0) + f(x-0))$$

for any $x \in [-\pi, \pi]$ pointwise. But since

$$\sigma_N f(x) = \frac{\sum_{j=0}^N S_j f(x)}{N+1}$$

then $S_N f(x)$ may converge (if it converges) only to the value

$$\frac{1}{2} (f(x+0) + f(x-0)).$$

We can obtain some sufficient conditions when the limit in (10.11) exists.

Theorem 15. *Suppose that $S(x)$ is chosen so that*

$$\int_0^\delta \frac{|f(x+y) + f(x-y) - 2S(x)|}{y} dy < \infty \tag{10.12}$$

pointwise or uniformly in $x \in [-\pi, \pi]$. Then

$$\lim_{N\to\infty} S_N f(x) = S(x) \tag{10.13}$$

pointwise or uniformly in $x \in [-\pi, \pi]$.

Proof. The claim follows immediately from (10.11) and Riemann-Lebesgue lemma. □

Remark. If in (10.13) we have uniform convergence then $S(x)$ must necessarily be periodic ($S(-\pi) = S(\pi)$) and continuous on the interval $[-\pi, \pi]$.

Corollary. *Suppose periodic f belongs to Hölder space $C^\alpha[-\pi, \pi]$ for some $0 < \alpha \le 1$. Then*

$$\lim_{N\to\infty} S_N f(x) = f(x)$$

uniformly in $x \in [-\pi, \pi]$.

Proof. Since $f \in C^\alpha[-\pi, \pi]$ then

$$|f(x+y) + f(x-y) - 2f(x)| \le |f(x+y) - f(x)| + |f(x-y) - f(x)| \le Cy^\alpha$$

for $0 < y < \delta$. It means that the condition (10.12) holds with $S(x) = f(x)$ uniformly in $x \in [-\pi, \pi]$. Thus, this corollary follows. □

Theorem 16 (Dirichlet, Jordan)**.** *Suppose that periodic f is of bounded variation on the interval $[x - \delta, x + \delta]$ for some $\delta > 0$ and some fixed x. Then*

$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}\left(f(x+0) + f(x-0)\right). \tag{10.14}$$

Proof. Since f is of bounded variation then the limit

$$\lim_{y\to 0+} \frac{1}{2}\left(f(x+y) + f(x-y)\right) = \frac{1}{2}\left(f(x+0) + f(x-0)\right) := S(x) \tag{10.15}$$

exists. For $0 < y < \delta$ we denote

$$F(y) := \frac{1}{2}\left(f(x+y) + f(x-y) - 2S(x)\right),$$

where $S(x)$ is defined by (10.15). Note that $F(0) = 0$. Let us also denote

$$G_N(y) := \int_0^y \frac{\sin(N + 1/2)t}{t} dt, \quad 0 < y \le \delta. \tag{10.16}$$

It is easy to check that

$$G_N(y) = \int_0^{(N+1/2)y} \frac{\sin\rho}{\rho} d\rho, \quad 0 < y \le \delta.$$

This representation implies that

$$\lim_{N\to\infty} G_N(y) = \int_0^\infty \frac{\sin\rho}{\rho} d\rho = \frac{\pi}{2}. \tag{10.17}$$

For fixed x we have from (10.11) and (10.16) that

$$S_N f(x) - S(x) = \frac{1}{\pi} \int_0^\delta F(y) G_N'(y) dy + o_N(1).$$

Here integration by parts gives

$$\begin{aligned} S_N f(x) - S(x) &= \frac{1}{\pi} \left(F(y) G_N(y)|_0^\delta - \int_0^\delta G_N(y) dF(y) \right) + o_N(1) \\ &= \frac{1}{\pi} \left(F(\delta) G_N(\delta) - \int_0^\delta G_N(y) dF(y) \right) + o_N(1), \end{aligned} \tag{10.18}$$

where the last integral is well-defined as the Stieltjes integral with respect to function $F(y)$ of bounded variation and continuous function $G_N(y)$. Since the limit (10.17) holds and $G_N(y)$ is continuous, we can consider the limit in (10.18) when $N \to \infty$. Hence, we obtain

$$\lim_{N\to\infty} (S_N f(x) - S(x)) = \frac{1}{\pi} \left(F(\delta) \frac{\pi}{2} - \frac{\pi}{2} \int_0^\delta dF(y) \right) = \frac{1}{2} (F(\delta) - F(\delta) + F(0)) = 0.$$

This finishes the proof. □

Corollary. *If f is periodic and belongs to Sobolev space $W_1^1(-\pi, \pi)$ then its trigonometric Fourier series converges pointwise to $f(x)$ everywhere.*

Proof. Since $f \in W_1^1(-\pi, \pi)$ then it is of bounded variation and continuous on the interval $[-\pi, \pi]$. In this case the value $S(x)$ from (10.15) equals $f(x)$ at any point $x \in [-\pi, \pi]$. Thus,

$$\lim_{N\to\infty} S_N f(x) = f(x)$$

pointwise in $x \in [-\pi, \pi]$. □

Remark. The above proof does not allow us to conclude uniform convergence of the trigonometric Fourier series of functions from Sobolev space $W_1^1(-\pi, \pi)$. However, uniform convergence is valid as we will prove later in this chapter.

Exercise 42. Show that

$$f(x) = \frac{1}{\log \frac{1}{|x|}}, \quad |x| < 1/2$$

is of bounded variation but this function does not satisfy condition (10.12) at $x = 0$.

Hint.

$$\left| \int_0^\delta \frac{1}{y \log y} dy \right| = +\infty.$$

Exercise 43. Show that

$$f(x) = x \sin \frac{1}{x}$$

satisfies condition (10.12) at $x = 0$ but this function is not of bounded variation.

We can return now to the question of term by term integration of the trigonometric Fourier series.

Theorem 17. *Suppose f belongs to $L^1(-\pi,\pi)$. Then*

$$\lim_{N\to\infty}\int_a^b S_N f(x)dx = \int_a^b f(x)dx$$

for any interval $(a,b)\subset[-\pi,\pi]$.

Proof. For a given L^1-function f (not necessarily periodic) introduce a new function F as

$$F(x) := \int_{-\pi}^{x}(f(t)-c_0(f))dt. \tag{10.19}$$

It is clear that $F(x)$ belongs to Sobolev space $W_1^1(-\pi,\pi)$ with $F(-\pi)=F(\pi)=0$ (periodicity) and

$$F'(x) = f(x) - c_0(f).$$

This implies

$$c_n(F') = inc_n(F) = c_n(f), \quad n\neq 0, \quad c_0(F')=0.$$

Corollary of Theorem 16 gives us that $F(x)$ has everywhere convergent trigonometric Fourier series

$$F(x) = c_0(F) + \sum_{n\neq 0}\frac{c_n(f)}{in}e^{inx}. \tag{10.20}$$

In particular, for any $-\pi\le a<b\le\pi$ we have from (10.20) that

$$F(b)-F(a) = \sum_{n\neq 0}\frac{c_n(f)}{in}(e^{inb}-e^{ina}) = \sum_{n\neq 0}c_n(f)\int_a^b e^{inx}dx$$

or equivalently (see (10.19)),

$$\begin{aligned}\int_{-\pi}^{b}(f(x)-c_0(f))dx - \int_{-\pi}^{a}(f(x)-c_0(f))dx &= \int_a^b f(x)dx - (b-a)c_0(f)\\ &= \sum_{n\neq 0}c_n(f)\int_a^b e^{inx}dx.\end{aligned}$$

Thus, we obtain finally

$$\int_a^b f(x)dx = \sum_{n=-\infty}^{\infty}c_n(f)\int_a^b e^{inx}dx = \lim_{N\to\infty}\sum_{|n|\le N}c_n(f)\int_a^b e^{inx}dx = \lim_{N\to\infty}\int_a^b S_N f(x)dx.$$

This proves the theorem. □

Exercise 44. Calculate $c_0(F)$ for F defined by (10.19).

Corollary 1. *If $f \in L^1(-\pi, \pi)$ then the series*

$$\sum_{n\neq 0} \frac{c_n(f)}{n} \quad \textit{and} \quad \sum_{n\neq 0} \frac{c_n(f)(-1)^n}{n} \tag{10.21}$$

converge.

Proof. Follows from (10.20). □

Corollary 2. *The series*

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{\log(1+n)}$$

is not a Fourier series of an L^1-function.

Proof. Let us assume on the contrary that there is $f \in L^1(-\pi, \pi)$ such that

$$\begin{aligned} f(x) \quad \sim \quad & \sum_{n=1}^{\infty} \frac{\sin(nx)}{\log(1+n)} = \frac{1}{2i} \sum_{n=1}^{\infty} \frac{e^{inx}}{\log(1+n)} - \frac{1}{2i} \sum_{n=1}^{\infty} \frac{e^{-inx}}{\log(1+n)} \\ = \quad & \frac{1}{2i} \sum_{n=1}^{\infty} \frac{e^{inx}}{\log(1+n)} + \frac{1}{2i} \sum_{n=-1}^{-\infty} \frac{\mathrm{sgn}(n) e^{inx}}{\log(1+|n|)} = \frac{1}{2i} \sum_{n\neq 0} \frac{\mathrm{sgn}(n) e^{inx}}{\log(1+|n|)} \end{aligned}$$

i.e. we have

$$c_n(f) = \frac{1}{2i} \frac{\mathrm{sgn}(n)}{\log(1+|n|)}, \quad n \neq 0, \quad c_0(f) = 0.$$

Since $c_n(f) = -c_{-n}(f)$ then this trigonometric Fourier series can be interpreted as the Fourier series of some odd L^1-function. Then Corollary 1 of Theorem 17 implies that

$$\sum_{n\neq 0} \frac{\mathrm{sgn}(n)}{2in \log(1+|n|)} = \frac{1}{i} \sum_{n=1}^{\infty} \frac{1}{n \log(1+n)}$$

must be convergent. But this is not true. This contradiction proves this corollary. □

Remark. If we define the function f by the series in Corollary 2 then it turns out that

$$\int_{-\pi}^{\pi} f(x) dx = 0, \quad \int_{-\pi}^{\pi} |f(x)| dx = +\infty.$$

Recall that the Poisson kernel is equal to

$$P_r(x) = \frac{1 - r^2}{1 - 2r\cos x + r^2}, \quad 0 \leq r < 1$$

and its trigonometric Fourier series is

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}.$$

The Corollary of Theorem 15 shows us that this series converges to $P_r(x)$ uniformly in $x \in [-\pi, \pi]$.

Theorem 18. *Suppose that $f \in C[-\pi,\pi]$ is periodic. Then*

$$\lim_{r\to 1-}(P_r * f)(x) = f(x) \tag{10.22}$$

uniformly in $x \in [-\pi,\pi]$ or

$$\lim_{r\to 1-}\sum_{n=-\infty}^{\infty} r^{|n|}c_n(f)e^{inx} = f(x) \tag{10.23}$$

uniformly in $x \in [-\pi,\pi]$ even if $f(x)$ has no convergent trigonometric Fourier series.

Proof. Using the normalization

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x)dx = 1$$

we have

$$\begin{aligned}(P_r * f)(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(y)(f(x-y)-f(x))dy \\ &= \frac{1}{2\pi}\int_{|y|\le\delta} P_r(y)(f(x-y)-f(x))dy \\ &+ \frac{1}{2\pi}\int_{\delta\le|y|\le\pi} P_r(y)(f(x-y)-f(x))dy := I_1 + I_2.\end{aligned}$$

Since f is continuous on $[-\pi,\pi]$ then I_1 can be estimated as

$$|I_1| \le \sup_{x\in[-\pi,\pi],|y|\le\delta}|f(x-y)-f(x)|\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(y)dy = \sup_{x\in[-\pi,\pi],|y|\le\delta}|f(x-y)-f(x)| \to 0$$

as $\delta \to 0$. At the same time, I_2 can be estimated as

$$|I_2| \le 2\max_{|x|\le\pi}|f(x)|\frac{1}{2\pi}\int_{\delta\le|y|\le\pi}P_r(y)dy.$$

For $\delta \le |y| \le \pi$ the Poisson kernel can be estimated as

$$P_r(y) = \frac{1-r^2}{1-2r\cos y + r^2} = \frac{1-r^2}{4r\sin^2 y/2 + (1-r)^2} \le \frac{1-r}{2r\sin^2 y/2} \le \frac{1-r}{2r\delta^2/\pi^2} = \frac{\pi^2}{2}\frac{1-r}{r\delta^2}.$$

If we choose $\delta^4 = 1-r$ then I_2 is estimated as

$$|I_2| \le \frac{1}{\pi}\max_{|x|\le\pi}|f(x)|\frac{\pi^2}{2}\frac{1-r}{r\sqrt{1-r}} = \frac{\pi}{2}\max_{|x|\le\pi}|f(x)|\frac{\sqrt{1-r}}{r} \to 0$$

as $r \to 1-$. Hence, the estimates for I_1 and I_2 show that

$$\lim_{r\to 1-}((P_r * f)(x) - f(x)) = 0$$

uniformly in $x \in [-\pi,\pi]$. The equality (10.23) follows from this fact and Exercise 12. This completes the proof. □

We will prove now the well-known Hardy's theorem and then apply it to the uniform convergence of the trigonometric sum $S_N f(x)$.

Theorem 19 (Hardy, 1949)**.** *Let* $\{a_k\}_{k=0}^{\infty}$ *be a sequence of complex numbers such that*

$$k|a_k| \le M, \quad k = 0, 1, 2, \dots, \tag{10.24}$$

where the constant M *is independent of* k*. If the limit*

$$\lim_{n\to\infty} \sigma_n := \lim_{n\to\infty} \sum_{j=0}^{n} \left(1 - \frac{j}{n+1}\right) a_j = a \tag{10.25}$$

exists then

$$\lim_{n\to\infty} (\sigma_n - s_n) = 0,$$

where $s_n = \sum_{j=0}^{n} a_j$*, i.e. also*

$$\lim_{n\to\infty} s_n = a. \tag{10.26}$$

If a_k *depends on* x *and (10.24) holds uniformly in* x *and convergence in (10.25) is uniform then convergence in (10.26) is also uniform.*

Proof. For $n < m$ it is true that

$$(m+1)\sigma_m - (n+1)\sigma_n - \sum_{j=n+1}^{m} (m+1-j)a_j = (m-n)s_n.$$

Indeed,

$$\begin{aligned}
&(m+1)\sigma_m - (n+1)\sigma_n - \sum_{j=n+1}^{m} (m+1-j)a_j \\
&\qquad = \sum_{j=0}^{m}(m+1-j)a_j - \sum_{j=0}^{n}(n+1-j)a_j - \sum_{j=n+1}^{m} (m+1-j)a_j \\
&\qquad = \sum_{j=0}^{n}(m+1-j)a_j - \sum_{j=0}^{n}(n+1-j)a_j = \sum_{j=0}^{n}(m-n)a_j = (m-n)s_n.
\end{aligned}$$

That's why,

$$(m+1)\sigma_m - (n+1)\sigma_n - \sum_{j=n+1}^{m} (m+1-j)a_j - (m-n)\sigma_n = (m-n)s_n - (m-n)\sigma_n$$

or, equivalently,

$$m(\sigma_m - \sigma_n) + (\sigma_m - \sigma_n) - \sum_{j=n+1}^{m} (m+1-j)a_j = (m-n)(s_n - \sigma_n)$$

i.e.

$$\frac{m+1}{m-n}(\sigma_m - \sigma_n) - \frac{m+1}{m-n}\sum_{j=n+1}^{m}\left(1-\frac{j}{m+1}\right)a_j = s_n - \sigma_n.$$

Let $m > n \to \infty$ such that $m/n \sim 1+\delta$ (i.e. $\lim_{m,n\to\infty} m/n = 1+\delta$) with some positive δ to be chosen. Since σ_m is a Cauchy sequence by (10.25) then

$$\frac{m+1}{m-n}(\sigma_m - \sigma_n) \to 0, \quad m > n \to \infty.$$

At the same time, the condition (10.24) implies that

$$\begin{aligned}
\left|\frac{m+1}{m-n}\sum_{j=n+1}^{m}\left(1-\frac{j}{m+1}\right)a_j\right| &\le \frac{m+1}{m-n}\sum_{j=n+1}^{m}\left(1-\frac{j}{m+1}\right)\frac{M}{j} \\
&\sim M(1+1/\delta)\sum_{j=n+1}^{m}\left(\frac{1}{j}-\frac{1}{m+1}\right) \\
&= M(1+1/\delta)\left(\sum_{j=n+1}^{m}\frac{1}{j}-\frac{m-n}{m+1}\right) \\
&\le M(1+1/\delta)\left(\int_n^m \frac{1}{\xi}d\xi - \frac{m-n}{m+1}\right) \\
&= M(1+1/\delta)\left(\log\frac{m}{n}-\frac{m-n}{m+1}\right) \\
&\sim M(1+1/\delta)\left(\log(1+\delta)-\frac{\delta}{1+\delta+1/n}\right) \\
&\sim M(1+1/\delta)\left(\delta-\delta^2/2+o(\delta^2)-\delta(1-\delta+\mathcal{O}(\delta^2))\right) \\
&= M(1+1/\delta)\left(\delta^2/2+o(\delta^2)\right) \le 2M\delta
\end{aligned}$$

if δ is chosen small enough. Since δ is arbitrary we may conclude that also the second term converges to zero. This proves the theorem. □

Corollary 1. *Suppose that $f \in C[-\pi,\pi]$ is periodic and that its Fourier coefficients satisfy*

$$|c_n(f)| \le \frac{M}{|n|}, \quad n \ne 0$$

with positive constant M which does not depend on n. Then the trigonometric Fourier series of f converges to f uniformly in $x \in [-\pi,\pi]$.

Proof. Since $f \in C[-\pi,\pi]$ is periodic then Theorem 3 gives the convergence of Fejér means

$$\lim_{N\to\infty}\sigma_N f(x) = f(x)$$

uniformly in $x \in [-\pi,\pi]$. Let us denote $a_0 = c_0(f)$ and

$$a_k(x) = c_k(f)e^{ikx} + c_{-k}(f)e^{-ikx}, \quad k = 1,2,\ldots.$$

Then

$$s_n \equiv \sum_{k=0}^{n} a_k(x) = S_N f(x)$$

and

$$\sigma_n f(x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k f(x) = \frac{1}{n+1} \sum_{k=0}^{n} s_k.$$

Thus, we are in the frame of Hardy's theorem because

$$|a_k(x)| \le \frac{2M}{k}, \quad k = 1, 2, \ldots.$$

Since this inequality is uniform in $x \in [-\pi, \pi]$ applying Hardy's theorem we obtain that

$$\lim_{N\to\infty} s_N \equiv \lim_{N\to\infty} S_N f(x) = f(x)$$

uniformly in $x \in [-\pi, \pi]$. □

Corollary 2. *If $f \in W_1^1(-\pi, \pi)$ is periodic then*

$$\lim_{N\to\infty} S_N f(x) = f(x)$$

uniformly in $x \in [-\pi, \pi]$.

Proof. Since $f \in W_1^1(-\pi, \pi)$ is periodic then there is $g \in L^1(-\pi, \pi)$ such that

$$f(x) = \int_{-\pi}^{x} g(t)dt + f(-\pi), \quad \int_{-\pi}^{\pi} g(t)dt = 0.$$

Thus, $f' = g$ and

$$c_n(g) = inc_n(f)$$

or, equivalently,

$$|c_n(f)| = \left| \frac{c_n(g)}{in} \right| \le \frac{M}{|n|}, \quad n \ne 0.$$

Due to imbedding (see Lemma 1.3) the function f is continuous on the interval $[-\pi, \pi]$. Using again Hardy's theorem we obtain

$$\lim_{N\to\infty} S_N f(x) = f(x)$$

uniformly in $x \in [-\pi, \pi]$. This finishes the proof. □

Let us return to some special trigonometric Fourier series. Namely, we consider functions $f_1(x)$ and $f_2(x)$ which are defined by the Fourier series

$$f_1(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n \log(1+n)} \tag{10.27}$$

and

$$f_2(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n\log(1+n)}. \tag{10.28}$$

These functions are well-defined for all $x \in [-\pi, \pi] \setminus \{0\}$, see Theorem 1. In addition, $f_1(0) = 0$ whereas $f_2(0)$ is not defined since the series (10.28) diverges at zero. We will show that $f_2(x)$ does not belong to $W_1^1(-\pi,\pi)$ but $f_1(x)$ does.

If we assume on the contrary that $f_2 \in W_1^1(-\pi,\pi)$ then its derivative f_2' has the Fourier series

$$f_2'(x) \sim -\sum_{n=1}^{\infty} \frac{\sin(nx)}{\log(1+n)}.$$

But due to Corollary 2 of Theorem 17 this is not a Fourier series of an L^1-function. This contradiction proves that $f_2 \notin W_1^1(-\pi,\pi)$.

Concerning the series (10.27) let us prove first that it converges uniformly in $x \in [-\pi,\pi]$ i.e. $f_1(x)$ is (at least) continuous on the interval $[-\pi,\pi]$. Indeed, by summation by parts we obtain for $0 \le M < N$ that

$$\begin{aligned}\sum_{n=M+1}^{N} \frac{\sin(nx)}{n\log(1+n)} &= \sum_{n=M+1}^{N} \frac{1}{n\log(1+n)}\left(\sum_{k=1}^{n}\sin(kx) - \sum_{k=1}^{n-1}\sin(kx)\right)\\ &= \frac{\sin(Nx)}{N\log(1+N)} - \frac{\sin(Mx)}{M\log(1+M)}\\ &\quad - \sum_{n=M+1}^{N}\left(\sum_{k=1}^{n}\sin(kx)\right)\left(\frac{1}{(n+1)\log(2+n)} - \frac{1}{n\log(1+n)}\right).\end{aligned} \tag{10.29}$$

Using the calculation from Exercise 15 we have

$$\sum_{k=1}^{n}\sin(kx) = \frac{\cos x/2 - \cos(n+1/2)x}{2\sin x/2} = \frac{\sin(nx/2)\sin((n+1)x/2)}{\sin x/2}. \tag{10.30}$$

The first two terms on the right-hand side of (10.29) converge to zero as $N > M \to \infty$ uniformly in $x \in [-\pi,\pi]$. The sum on the right-hand side of (10.29) becomes, using (10.30),

$$\begin{aligned}-\sum_{n=M+1}^{N} &\frac{\sin(nx/2)\sin((n+1)x/2)}{\sin x/2}\,\frac{n\log\frac{2+n}{1+n} + \log(2+n)}{n(n+1)\log(1+n)\log(2+n)}\\ &= -\sum_{n=M+1}^{N}\frac{\sin(nx/2)\sin((n+1)x/2)}{\sin x/2}\,\frac{\log\frac{2+n}{1+n}}{(n+1)\log(1+n)\log(2+n)}\\ &\quad - \sum_{n=M+1}^{N}\frac{\sin(nx/2)\sin((n+1)x/2)}{\sin x/2}\,\frac{1}{n(n+1)\log(1+n)} = I_1 + I_2.\end{aligned}$$

Let us consider two cases: $n|x| < 1$ and $n|x| > 1$. In the first case,

$$\left|\frac{\sin(nx/2)\sin((n+1)x/2)}{\sin x/2}\right| \le \frac{n|x|/2\cdot 1}{|x|/\pi} = \frac{\pi n}{2}.$$

Then

$$\begin{aligned}
|I_1| &\le \frac{\pi}{2}\sum_{n=M+1}^{\left[\frac{1}{|x|}\right]} \frac{n\log\left(1+1/(n+1)\right)}{(n+1)\log(1+n)\log(2+n)} < \frac{\pi}{2}\sum_{n=M+1}^{\left[\frac{1}{|x|}\right]} \frac{n\frac{1}{n}}{n\log^2 n} \\
&< \frac{\pi}{2}\sum_{n=M+1}^{\infty} \frac{1}{n\log^2 n} \to 0 \qquad (10.31)
\end{aligned}$$

as $M \to \infty$ uniformly in x. In the first case for I_2 we have, by integration by parts, that

$$\begin{aligned}
|I_2| &\le \frac{\pi}{2}\sum_{n=M+1}^{\left[\frac{1}{|x|}\right]} \frac{n\frac{n+1}{2}|x|}{n(n+1)\log(1+n)} = \frac{\pi}{4}|x|\sum_{n=M+1}^{\left[\frac{1}{|x|}\right]} \frac{1}{\log(1+n)} < \frac{\pi}{4}|x|\int_{M+1}^{1/|x|}\frac{dt}{\log t} \\
&= \frac{\pi}{4}|x|\left(\left.\frac{t}{\log t}\right|_{M+1}^{1/|x|} + \int_{M+1}^{1/|x|}\frac{dt}{\log^2 t}\right) \\
&= \frac{\pi}{4}|x|\left(\frac{1/|x|}{\log(1/|x|)} - \frac{M+1}{\log(M+1)} + \int_{M+1}^{1/|x|}\frac{dt}{\log^2 t}\right) \\
&\le \frac{\pi}{4}\left(\frac{1}{\log(1/|x|)} + \frac{|x|(M+1)}{\log(M+1)} + \frac{|x|}{\log^2(M+1)}(1/|x| + M+1)\right) \\
&< \frac{\pi}{4}\left(\frac{1}{\log(M+1)} + \frac{1}{\log(M+1)} + \frac{1}{\log^2(M+1)} + \frac{1}{\log^2(M+1)}\right) \\
&\le \frac{\pi}{\log(M+1)} \to 0 \qquad (10.32)
\end{aligned}$$

uniformly in x as $M \to \infty$. In the second case,

$$\left|\frac{\sin(nx/2)\sin((n+1)x/2)}{\sin x/2}\right| \le \frac{1}{|\sin x/2|} \le \frac{\pi}{|x|} \le \pi n.$$

Then

$$\begin{aligned}
|I_1| &\le \pi\sum_{n=\left[\frac{1}{|x|}\right]}^{N} \frac{n\log\left(1+1/(n+1)\right)}{(n+1)\log(1+n)\log(2+n)} < \pi\sum_{n=\left[\frac{1}{|x|}\right]}^{N} \frac{n\frac{1}{n}}{n\log^2 n} \\
&< \pi\sum_{n=\left[\frac{1}{|x|}\right]\ge M}^{\infty} \frac{1}{n\log^2 n} \to 0, \quad M\to\infty \qquad (10.33)
\end{aligned}$$

uniformly in x. For I_2 we have, by integration by parts,

$$\begin{aligned}
|I_2| &\leq \frac{1}{|\sin x/2|}\sum_{n=[\frac{1}{|x|}]+1}^{N}\frac{1}{n^2\log n} \leq \pi\frac{1}{|x|}\int_{[\frac{1}{|x|}]\geq M}^{\infty}\frac{dt}{t^2\log t} \\
&= \pi\frac{1}{|x|}\left(-\frac{1}{t\log t}\Big|_{[\frac{1}{|x|}]\geq M}^{\infty} + \int_{[\frac{1}{|x|}]\geq M}^{\infty}\frac{dt}{t^2\log^2 t}\right) \\
&= \pi\left(\frac{1/|x|}{[1/|x|]\log[1/|x|]} + \frac{1}{|x|}\int_{[\frac{1}{|x|}]\geq M}^{\infty}\frac{dt}{t^2\log^2 t}\right) \\
&< \pi\left(\frac{[1/|x|]+1}{[1/|x|]\log[1/|x|]} + \frac{[1/|x|]+1}{[1/|x|]}\int_{M}^{\infty}\frac{dt}{t\log^2 t}\right) \\
&\leq \pi\left(\frac{1+1/M}{\log M} + \frac{1+1/M}{\log M}\right) \to 0, \quad M\to\infty \qquad (10.34)
\end{aligned}$$

uniformly in x. Finally, we may conclude that the trigonometric Fourier series (10.27) converges uniformly on the interval $[-\pi,\pi]$ and therefore it defines continuous function $f_1(x)$. This series, as well as (10.28), can be differentiated term by term for $\pi \geq |x| \geq \delta > 0$ because the series

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{\log(1+n)}$$

converges uniformly (see Corollary of Theorem 1) for $\pi \geq |x| \geq \delta > 0$. It means that for this interval $f_1(x)$ belongs to C^1. Thus, it remains to investigate the behaviour of the series (10.27) when $x \to 0+$. But the estimates (10.31)-(10.34) show us that (if we choose $M \asymp 1/x, x\to 0+$) the function $f_1(x)$ from (10.27) has the asymptotic behaviour

$$f_1(x) \sim \frac{C}{\log x}. \qquad (10.35)$$

It is possible to prove (see Zigmund: Trigonometric series, Vol. I, Chapter V, formula (2.19)) that the asymptotic (10.35) can be differentiated and we obtain

$$f_1'(x) \sim -\frac{C}{x\log^2 x}.$$

This singularity is integrable at zero. Thus, the function $f_1(x)$ belongs to $W_1^1(-\pi,\pi)$.

Exercise 45. Prove that the series (10.27) and (10.28) do not converge absolutely.

11 Formulation of discrete Fourier transform and its properties.

Let $x(t)$ be 2π-periodic continuous signal. Assume that $x(t)$ can be represented by an absolutely convergent trigonometric Fourier series

$$x(t) = \sum_{m=-\infty}^{\infty} c_m e^{imt}, \quad t \in [-\pi, \pi], \tag{11.1}$$

where c_m are the Fourier coefficients of $x(t)$.

Let now N be an even positive integer and

$$t_k = \frac{2\pi k}{N}, \quad k = -\frac{N}{2}, \dots, \frac{N}{2} - 1.$$

Then $x(t_k)$ is a response at t_k i.e.

$$x(t_k) = \sum_{m=-\infty}^{\infty} c_m e^{i\frac{2\pi km}{N}}. \tag{11.2}$$

Since $e^{i2\pi kl} = 1$ for integers k and l then the series (11.2) can be rewritten as

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} c_m e^{i\frac{2\pi k}{N}(m-lN)} &= \sum_{l=-\infty}^{\infty} \sum_{-N/2 \le m-lN \le N/2-1} c_m e^{i\frac{2\pi k}{N}(m-lN)} \\
&= \sum_{l=-\infty}^{\infty} \sum_{n=-N/2}^{N/2-1} c_{n+lN} e^{i\frac{2\pi k}{N}n} \\
&= \sum_{n=-N/2}^{N/2-1} e^{i\frac{2\pi k}{N}n} \sum_{l=-\infty}^{\infty} c_{n+lN} = \sum_{n=-N/2}^{N/2-1} e^{i\frac{2\pi k}{N}n} X_n,
\end{aligned} \tag{11.3}$$

where $X_n, n = -N/2, \dots, N/2 - 1$ is given by

$$X_n = \sum_{l=-\infty}^{\infty} c_{n+lN}. \tag{11.4}$$

In the calculation (11.3) we have used the fact that the series (11.1) converges absolutely. Combining (11.2) and (11.3) we obtain

$$x_k := x(t_k) = \sum_{n=-N/2}^{N/2-1} X_n e^{i\frac{2\pi kn}{N}}. \tag{11.5}$$

The formula (11.5) can be viewed as an inverse discrete Fourier transform and it appeared quite naturally in the discretization of a continuous periodic signal. Moreover, the formula (11.4) becomes the main property of this approach. Since

$$\sum_{n=-N/2}^{N/2-1} e^{i(k-m)\frac{2\pi n}{N}} = \begin{cases} 0, & k-m \neq 0, \pm N, \pm 2N, \dots \\ N, & k-m = 0, \pm N, \pm 2N, \dots \end{cases} \tag{11.6}$$

then we can solve the linear system (11.5) with respect to X_n for $n = -N/2, \dots, N/2-1$ and obtain

$$X_n = \frac{1}{N} \sum_{k=-N/2}^{N/2-1} x_k e^{-i\frac{2\pi kn}{N}}. \tag{11.7}$$

Exercise 46. Prove (11.6) and (11.7).

Actually, the formulas (11.5) and (11.7) give us the inverse and direct discrete Fourier transforms, respectively.

Definition 11.1. The sequence $\{X_n\}_{n=-N/2}^{N/2-1}$ of complex numbers is called the *discrete Fourier transform* (DFT) of the sequence $\{Y_k\}_{k=-N/2}^{N/2-1}$ if for each $n = -N/2, \dots, N/2-1$ we have

$$X_n = \frac{1}{N} \sum_{k=-N/2}^{N/2-1} Y_k e^{-i\frac{2\pi kn}{N}}. \tag{11.8}$$

We use the symbol $\mathcal{F}$ for DFT and write

$$X_n = \mathcal{F}(Y_k)_n$$

or simply $X = \mathcal{F}(Y)$.

Definition 11.2. The sequence $\{Z_k\}_{k=-N/2}^{N/2-1}$ of complex numbers is called the *inverse discrete Fourier transform* (IDFT) of the sequence $\{X_n\}_{n=-N/2}^{N/2-1}$ if for each $k = -N/2, \dots, N/2-1$ we have

$$Z_k = \sum_{n=-N/2}^{N/2-1} X_n e^{i\frac{2\pi kn}{N}}. \tag{11.9}$$

We use the symbol $\mathcal{F}^{-1}$ for IDFT and write

$$Z_k = \mathcal{F}^{-1}(X_n)_k$$

or simply $Z = \mathcal{F}^{-1}(X)$.

The properties of DFT and IDFT are collected in the following lemmas.

Lemma 11.1. *The following equalities hold:*

1) $\mathcal{F}^{-1}(\mathcal{F}(Y)) = Y$*;*

2) $\mathcal{F}(\mathcal{F}^{-1}(X)) = X$*;*

3)

$$\sum_{k=-N/2}^{N/2-1} \mathcal{F}(X_n)_k \overline{\mathcal{F}(Y_n)_k} = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} X_n \overline{Y_n}.$$

Proof. Using (11.6), (11.8) and (11.9) we have

$$\begin{aligned}
\mathcal{F}^{-1}(\mathcal{F}(Y))_k &= \sum_{n=-N/2}^{N/2-1} \mathcal{F}(Y_l)_n e^{i\frac{2\pi kn}{N}} = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} \left(\sum_{l=-N/2}^{N/2-1} Y_l e^{-i\frac{2\pi ln}{N}} \right) e^{i\frac{2\pi kn}{N}} \\
&= \frac{1}{N} \sum_{l=-N/2}^{N/2-1} Y_l \left(\sum_{n=-N/2}^{N/2-1} e^{i\frac{2\pi n(k-l)}{N}} \right) = \frac{1}{N} Y_k N = Y_k.
\end{aligned}$$

This proves part 1). Part 2) can be proved by the same manner. □

Exercise 47. Prove part 3) of Lemma 11.1.

Corollary 1 (Parseval equality)**.**

$$\frac{1}{N} \sum_{n=-N/2}^{N/2-1} |X_n|^2 = \sum_{k=-N/2}^{N/2-1} |\mathcal{F}(X_n)_k|^2.$$

Remark. Due to periodicity of the complex exponential we may extend the values of $X_m, m = -N/2, \dots, N/2-1$ periodically to any integer by

$$X_{m+lN} = X_m, \quad l = 0, \pm 1, \pm 2, \dots. \tag{11.10}$$

Corollary 2. *For sequence* $X = \{X_n\}_{n=-N/2}^{N/2-1}$ *we define*

$$X_{rev} = \{X_{N-n}\}_{n=-N/2}^{N/2-1}.$$

Then

$$\mathcal{F}^{-1}(X) = N\mathcal{F}(X_{rev}).$$

Proof. By Definition 11.2 and periodicity condition (11.10) we have

$$\begin{aligned}
N\mathcal{F}(X_{\text{rev}})_k &= \sum_{n=-N/2}^{N/2-1} X_{N-n} e^{-i\frac{2\pi kn}{N}} = \sum_{n=-N/2}^{N/2-1} X_{-n} e^{-i\frac{2\pi kn}{N}} = \sum_{n=N/2}^{-N/2+1} X_n e^{i\frac{2\pi kn}{N}} \\
&= \sum_{n=-N/2+1}^{N/2} X_n e^{i\frac{2\pi kn}{N}} = \sum_{n=-N/2}^{N/2-1} X_n e^{i\frac{2\pi kn}{N}} + X_{N/2} e^{i\pi k} - X_{-N/2} e^{-i\pi k} \\
&= \sum_{n=-N/2}^{N/2-1} X_n e^{i\frac{2\pi kn}{N}} = \mathcal{F}^{-1}(X)_k.
\end{aligned}$$

□

Definition 11.3. The *convolution* of sequences $X = \{X_n\}_{n=-N/2}^{N/2-1}$ and $Y = \{Y_n\}_{n=-N/2}^{N/2-1}$ is defined as the sequence whose elements are given by

$$(X * Y)_k = \sum_{l=-N/2}^{N/2-1} X_l Y_{k-l}, \tag{11.11}$$

where X_n and Y_n satisfy the periodicity condition (11.10).

Proposition. *For any integer l it is true that*

$$\sum_{m=-N/2-l}^{N/2-l-1} Y_m e^{-i\frac{2\pi nm}{N}} = \sum_{m=-N/2}^{N/2-1} Y_m e^{-i\frac{2\pi nm}{N}}.$$

Proof. The claim is trivial for $l = 0$. If $l > 0$ then

$$\begin{aligned}\sum_{m=-N/2-l}^{N/2-l-1} Y_m e^{-i\frac{2\pi nm}{N}} &= \sum_{m=-N/2}^{N/2-1} Y_m e^{-i\frac{2\pi nm}{N}} + \sum_{m=-N/2-l}^{-N/2-1} Y_m e^{-i\frac{2\pi nm}{N}} - \sum_{m=N/2-l}^{N/2-1} Y_m e^{-i\frac{2\pi nm}{N}} \\ &= \sum_{m=-N/2}^{N/2-1} Y_m e^{-i\frac{2\pi nm}{N}} + \sum_{m=N/2-l}^{N/2-1} Y_{m-N} e^{-i\frac{2\pi n}{N}(m-N)} \\ &- \sum_{m=N/2-l}^{N/2-1} Y_m e^{-i\frac{2\pi nm}{N}} = \sum_{m=-N/2}^{N/2-1} Y_m e^{-i\frac{2\pi nm}{N}}\end{aligned}$$

due to periodicity condition (11.10). If $l < 0$ then

$$\begin{aligned}\sum_{m=-N/2-l}^{N/2-l-1} Y_m e^{-i\frac{2\pi nm}{N}} &= \sum_{m=-N/2}^{N/2-1} Y_m e^{-i\frac{2\pi nm}{N}} + \sum_{m=N/2}^{N/2-l-1} Y_m e^{-i\frac{2\pi nm}{N}} - \sum_{m=-N/2}^{-N/2-l-1} Y_m e^{-i\frac{2\pi nm}{N}} \\ &= \sum_{m=-N/2}^{N/2-1} Y_m e^{-i\frac{2\pi nm}{N}} + \sum_{m=N/2}^{N/2-l-1} Y_m e^{-i\frac{2\pi nm}{N}} \\ &- \sum_{m=N/2}^{N/2-l-1} Y_{m-N} e^{-i\frac{2\pi n}{N}(m-N)} = \sum_{m=-N/2}^{N/2-1} Y_m e^{-i\frac{2\pi nm}{N}}\end{aligned}$$

due to periodicity condition (11.10). This proves the Proposition. □

Corollary. *For any integer l it is true that*

$$\sum_{m=-N/2-l}^{N/2-l-1} Y_m = \sum_{m=-N/2}^{N/2-1} Y_m.$$

Lemma 11.2. *The convolution (11.11) is symmetric i.e.*

$$(X * Y)_k = (Y * X)_k$$

for any $k = -N/2, \dots, N/2 - 1$.

Proof. We have

$$\begin{aligned}(X * Y)_k &= \sum_{l=-N/2}^{N/2-1} X_l Y_{k-l} = \sum_{j=k+N/2}^{k+1-N/2} Y_j X_{k-j} \\ &= \sum_{j=-N/2+(k+1)}^{N/2-1+(k+1)} Y_j X_{k-j} = \sum_{j=-N/2}^{N/2-1} Y_j X_{k-j} = (Y * X)_k\end{aligned}$$

by the above Corollary. □

Lemma 11.3. *For each* $n = -N/2, \dots, N/2 - 1$ *it is true that*

1) $\mathcal{F}(X * Y)_n = N\mathcal{F}(X)_n \mathcal{F}(Y)_n$;
2) $\mathcal{F}^{-1}(X * Y)_n = \mathcal{F}^{-1}(X)_n \mathcal{F}^{-1}(Y)_n$.

Proof. Using (11.11) we have

$$\begin{aligned}\mathcal{F}(X * Y)_n &= \frac{1}{N} \sum_{k=-N/2}^{N/2-1} (X * Y)_k e^{-i\frac{2\pi kn}{N}} = \frac{1}{N} \sum_{l=-N/2}^{N/2-1} X_l \sum_{k=-N/2}^{N/2-1} Y_{k-l} e^{-i\frac{2\pi kn}{N}} \\ &= \frac{1}{N} \sum_{l=-N/2}^{N/2-1} X_l \sum_{m=-N/2-l}^{N/2-l-1} Y_m e^{-i\frac{2\pi n}{N}(m+l)} \\ &= \frac{1}{N} \sum_{l=-N/2}^{N/2-1} X_l e^{-i\frac{2\pi nl}{N}} \sum_{m=-N/2-l}^{N/2-l-1} Y_m e^{-i\frac{2\pi nm}{N}}. \end{aligned} \tag{11.12}$$

The above Proposition allows us to rewrite (11.12) as

$$\mathcal{F}(X * Y)_n = \frac{1}{N} \sum_{l=-N/2}^{N/2-1} X_l e^{-i\frac{2\pi nl}{N}} \sum_{m=-N/2}^{N/2-1} Y_m e^{-i\frac{2\pi nm}{N}} = N\mathcal{F}(X)_n \mathcal{F}(Y)_n.$$

Part 2) is proved in a similar manner. □

Corollary. *For each* $n = -N/2, \dots, N/2 - 1$ *it is true that*

$$\mathcal{F}^{-1}(X \cdot Y)_n = \frac{1}{N} \left(\mathcal{F}^{-1}(X) * \mathcal{F}^{-1}(Y)\right)_n, \tag{11.13}$$

where $X \cdot Y$ *denotes the sequence* $\{X_k \cdot Y_k\}_{k=-N/2}^{N/2-1}$.

Proof. Lemmas 11.1 and 11.3 imply that

$$\left(\widetilde{X} * \widetilde{Y}\right)_n = \mathcal{F}^{-1}\left(\mathcal{F}\left(\widetilde{X} * \widetilde{Y}\right)\right)_n = N\mathcal{F}^{-1}\left(\mathcal{F}(\widetilde{X}) \cdot \mathcal{F}(\widetilde{Y})\right)_n .$$

Denoting $\widetilde{X} := \mathcal{F}^{-1}(X)$ and $\widetilde{Y} := \mathcal{F}^{-1}(Y)$ we obtain easily from the latter equality that

$$N\mathcal{F}^{-1}(X \cdot Y)_n = \left(\mathcal{F}^{-1}(X) * \mathcal{F}^{-1}(Y)\right)_n .$$

This finishes the proof. □

Let us return to the continuous signal $x(t), t \in [-\pi, \pi]$, which is represented by an absolutely convergent trigonometric Fourier series (11.1). Formula (11.4) allows us to obtain

$$\begin{aligned}\sum_{n=-N/2}^{N/2-1} |X_n - c_n| &= \sum_{n=-N/2}^{N/2-1} \left| \sum_{l=-\infty}^{\infty} c_{n+lN} - c_n \right| = \sum_{n=-N/2}^{N/2-1} \left| \sum_{l \neq 0} c_{n+lN} \right| \\ &\leq \sum_{n=-N/2}^{N/2-1} \sum_{l \neq 0} |c_{n+lN}| \leq \sum_{|\nu| \geq N/2} |c_\nu|. \end{aligned} \tag{11.14}$$

Similarly, we have

$$\begin{aligned}\left| \mathcal{F}^{-1}(X_n)_k - \sum_{n=-N/2}^{N/2-1} c_n e^{i\frac{2\pi kn}{N}} \right| &= \left| \sum_{n=-N/2}^{N/2-1} (X_n - c_n) e^{i\frac{2\pi kn}{N}} \right| \\ &\leq \sum_{n=-N/2}^{N/2-1} |X_n - c_n| \leq \sum_{|\nu| \geq N/2} |c_\nu|. \end{aligned} \tag{11.15}$$

In the formulas (11.14) and (11.15) the numbers c_n are the Fourier coefficients of the signal $x(t)$ and $\{X_n\}_{n=-N/2}^{N/2-1}$ is the DFT of $\{x(t_k)\}_{k=-N/2}^{N/2-1}$ with $t_k = 2\pi k/N$.

Theorem 20. *If $x(t)$ is periodic and belongs to Sobolev space $W_2^m(-\pi, \pi)$ for some $m = 1, 2, \ldots$ then*

$$X_n = c_n + o\left(\frac{1}{N^{m-1/2}}\right) \tag{11.16}$$

and

$$\mathcal{F}^{-1}(X_n)_k = \sum_{n=-N/2}^{N/2-1} c_n e^{i\frac{2\pi kn}{N}} + o\left(\frac{1}{N^{m-1/2}}\right) \tag{11.17}$$

uniformly in n and k from the set $\{-N/2, \ldots, N/2-1\}$.

Proof. Using Hölder's inequality we have

$$\sum_{|\nu|\geq N/2} |c_\nu| \leq \left(\sum_{|\nu|\geq N/2} |\nu|^{2m} |c_\nu|^2 \right)^{1/2} \left(\sum_{|\nu|\geq N/2} |\nu|^{-2m} \right)^{1/2}.$$

The first sum in the right hand side tends to zero as $N \to \infty$ due to Parseval equality for function from Sobolev space $W_2^m(-\pi, \pi)$. The second sum can be estimated precisely. Namely, since for any $m = 1, 2, \ldots$ we have

$$\left(\sum_{|\nu|\geq N/2} |\nu|^{-2m} \right)^{1/2} \asymp \left(\int_{N/2}^{\infty} t^{-2m} dt \right)^{1/2} \asymp N^{-m+1/2}$$

then (11.16) and (11.17) follow from the last estimate and (11.14) and (11.15), respectively. $\square$

Corollary. *An unknown periodic function $f \in W_2^m(-\pi, \pi), m = 1, 2, \ldots$ can be recovered from its IDFT as*

$$f\left(\frac{2\pi k}{N}\right) = \mathcal{F}^{-1}(X)_k + o\left(\frac{1}{N^{m-1/2}}\right)$$

uniformly in k from the set $\{-N/2, \ldots, N/2 - 1\}$.

Exercise 48. 1) Show that

$$\mathcal{F}\left(\{a^k\}_{k=-N/2}^{N/2-1}\right)_n = \begin{cases} 1, & a = e^{i\frac{2\pi n}{N}} \\ \frac{1}{N}(-1)^n \frac{a^{-N/2} - a^{N/2}}{1 - ae^{-i\frac{2\pi n}{N}}}, & a \neq e^{i\frac{2\pi n}{N}} \end{cases}$$

2) If sequence $Y = \{Y_k\}_{k=-N/2}^{N/2-1}$ is real then show that

$$\mathcal{F}(Y)_n = \overline{\mathcal{F}(Y)_{N-n}}.$$

12 Connection between discrete Fourier transform and Fourier transform.

If function $f(x)$ is integrable over the whole line i.e.

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty$$

then its *Fourier transform* is defined as

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx. \tag{12.1}$$

Similarly, *inverse Fourier transform* of an integrable function $g(\xi)$ is defined as

$$\mathcal{F}^{-1}g(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi)e^{ix\xi}d\xi. \tag{12.2}$$

Theorem 21 (Riemann-Lebesgue lemma)**.** *For any integrable function $f(x)$ its Fourier transform $\mathcal{F}f(\xi)$ is continuous and*

$$\lim_{\xi \to \pm\infty} \mathcal{F}f(\xi) = 0.$$

Proof. Since $e^{i\pi} = -1$ then we have

$$\widehat{f}(\xi) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\xi+i\pi}dx = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y+\pi/\xi)e^{-i\xi y}dy.$$

This fact implies that

$$-2\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f(x+\pi/\xi) - f(x))e^{-i\xi x}dx.$$

Hence

$$2|\widehat{f}(\xi)| \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x+\pi/\xi) - f(x)|dx \to 0$$

as $|\xi| \to \infty$ since f is integrable on the whole line. This is a well-known property of integrable functions. Continuity of $\widehat{f}(\xi)$ follows from the representation

$$\widehat{f}(\xi+h) - \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\xi}(e^{-ixh} - 1)dx$$

and its consequence

$$|\widehat{f}(\xi+h) - \widehat{f}(\xi)| \le \frac{1}{\sqrt{2\pi}} \int_{|xh|<\delta} |f(x)||e^{-ixh} - 1|dx + \frac{2}{\sqrt{2\pi}} \int_{|xh|>\delta} |f(x)|dx := I_1 + I_2.$$

For the first term I_1 we have the estimate

$$I_1 \le \frac{1}{\sqrt{2\pi}} \int_{|xh|<\delta} |f(x)||xh|dx < \frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|dx \to 0, \quad \delta \to 0.$$

For the second term I_2 we have

$$I_2 = \frac{2}{\sqrt{2\pi}} \int_{|x|>\delta/|h|} |f(x)|dx \to 0$$

as $|h| \to 0$. If we choose $\delta = |h|^{1/2}$ then both I_1 and I_2 tend to zero as $|h| \to 0$. This finishes the proof. □

If function $f(x)$ has integrable derivatives $f^{(k)}(x)$ of order $k = 0, 1, 2, \ldots, m$ then we say that f belongs to Sobolev space $W_1^m(\mathbb{R})$.

Exercise 49. Prove that if $f \in W_1^1(\mathbb{R})$ then $\lim_{x\to\pm\infty} f(x) = 0$.

Theorem 22 (Fourier inversion formula)**.** *Suppose that f belongs to $W_1^1(\mathbb{R})$. Then*

$$\mathcal{F}^{-1}(\mathcal{F}f)(x) = f(x)$$

at any point $x \in \mathbb{R}$.

Proof. First we prove that

$$\int_{-\infty}^{\infty} f(x)\widehat{g}(x)dx = \int_{-\infty}^{\infty} \widehat{f}(\xi)g(\xi)d\xi$$

for any two integrable functions f and g. Indeed,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\widehat{g}(x)dx &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} g(\xi)e^{-ix\xi}d\xi dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dxd\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)g(\xi)d\xi\end{aligned}$$

by Fubini's theorem. Suppose now that $g(\xi)$ is given by

$$g(\xi) = \begin{cases} 1, & |\xi| < n \\ 0, & |\xi| > n. \end{cases}$$

Its Fourier transform is equal to

$$\widehat{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{-ix\xi}d\xi = \frac{1}{\sqrt{2\pi}} \left.\frac{e^{-ix\xi}}{-ix}\right|_{-n}^{n} = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-ixn}}{-ix} - \frac{e^{ixn}}{-ix} \right) = \sqrt{\frac{2}{\pi}} \frac{\sin(nx)}{x}.$$

Thus, we have the equality

$$\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x)\frac{\sin(nx)}{x}dx = \int_{-n}^{n} \widehat{f}(\xi)d\xi,$$

where $f \in W_1^1(\mathbb{R})$. Letting $n \to \infty$ we obtain

$$\text{p.v.} \int_{-\infty}^{\infty} \widehat{f}(\xi) d\xi = \lim_{n\to\infty} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x) \frac{\sin(nx)}{x} dx. \tag{12.3}$$

We will prove that the limit in (12.3) is actually equal to $\sqrt{2\pi} f(0)$. Since

$$\int_{-\infty}^{\infty} \frac{\sin(nx)}{x} dx = \pi$$

then the limit in (12.3) can be rewritten as

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x) \frac{\sin(nx)}{x} dx &= \sqrt{2\pi} f(0) + \lim_{n\to\infty} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} (f(x) - f(0)) \frac{\sin(nx)}{x} dx \\
&= \sqrt{2\pi} f(0) + \lim_{n\to\infty} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} (f(t/n) - f(0)) \frac{\sin(t)}{t} dt.
\end{aligned}$$

It remains to show that the latter limit is equal to zero. In order to prove this fact we split

$$\begin{aligned}
&\int_{-\infty}^{\infty} (f(t/n) - f(0)) \frac{\sin(t)}{t} dt \\
&\qquad = \int_{|t|<1} (f(t/n) - f(0)) \frac{\sin(t)}{t} dt + \int_{|t|>1} (f(t/n) - f(0)) \frac{\sin(t)}{t} dt := I_1 + I_2.
\end{aligned}$$

Since $f \in W_1^1(\mathbb{R})$ then it can be proved that f is continuous. Therefore

$$|I_1| \le \sup_{|t|<1} |f(t/n) - f(0)| \int_{|t|<1} \left| \frac{\sin(t)}{t} \right| dt \to 0, \quad n \to \infty.$$

For the second term I_2 we first change variables back as

$$\begin{aligned}
I_2 &= \int_{|z|>1/n} (f(z) - f(0)) \frac{\sin(nz)}{z} dz \\
&= \int_{\frac{1}{n}}^{\infty} (f(z) - f(0)) \left(\int_0^z \frac{\sin(ny)}{y} dy \right)' dz + \int_{-\infty}^{-\frac{1}{n}} (f(z) - f(0)) \left(\int_0^z \frac{\sin(ny)}{y} dy \right)' dz.
\end{aligned}$$

Integration by parts in these integrals leads to

$$\begin{aligned}
I_2 &= (f(z)-f(0))\int_0^z \frac{\sin(ny)}{y}dy\Big|_{1/n}^{\infty} - \int_{1/n}^{\infty} f'(z)\int_0^z \frac{\sin(ny)}{y}dydz \\
&+ (f(z)-f(0))\int_0^z \frac{\sin(ny)}{y}dy\Big|_{-\infty}^{-1/n} - \int_{-\infty}^{-1/n} f'(z)\int_0^z \frac{\sin(ny)}{y}dydz \\
&= (\lim_{z\to\infty} f(z)-f(0))\int_0^{\infty} \frac{\sin(ny)}{y}dy - (f(1/n)-f(0))\int_0^{1/n} \frac{\sin(ny)}{y}dy \\
&- \int_{1/n}^{\infty} f'(z)\int_0^{nz} \frac{\sin(t)}{t}dtdz \\
&+ (f(-1/n)-f(0))\int_0^{-1/n} \frac{\sin(ny)}{y}dy - (\lim_{z\to-\infty} f(z)-f(0))\int_0^{-\infty} \frac{\sin(ny)}{y}dy \\
&- \int_{-\infty}^{-1/n} f'(z)\int_0^{nz} \frac{\sin(t)}{t}dtdz.
\end{aligned}$$

Since $\lim_{z\to\pm\infty} f(z) = 0$ (see Exercise 49) and since f is continuous we obtain (when $n \to \infty$)

$$\begin{aligned}
I_2 &\to -f(0)\frac{\pi}{2} - \lim_{n\to\infty}\int_{1/n}^{\infty} f'(z)\int_0^{nz} \frac{\sin(t)}{t}dtdz \\
&- f(0)\frac{\pi}{2} - \lim_{n\to\infty}\int_{-\infty}^{-1/n} f'(z)\int_0^{nz} \frac{\sin(t)}{t}dtdz \\
&= -\pi f(0) - \lim_{n\to\infty}\int_{1/n}^{\infty} f'(z)\int_0^{nz} \frac{\sin(t)}{t}dtdz - \lim_{n\to\infty}\int_{-\infty}^{-1/n} f'(z)\int_0^{nz} \frac{\sin(t)}{t}dtdz \\
&= -\pi f(0) - \int_0^{\infty} f'(z)\frac{\pi}{2}dz + \int_{-\infty}^{0} f'(z)\frac{\pi}{2}dz \\
&= -\pi f(0) + f(0)\frac{\pi}{2} + f(0)\frac{\pi}{2} = 0.
\end{aligned}$$

Here we used again the fact that $\lim_{z\to\pm\infty} f(z) = 0$ and Lebesgue's dominated convergence theorem. Thus, (12.3) transforms to

$$\text{p.v.}\int_{-\infty}^{\infty} \widehat{f}(\xi)d\xi = \sqrt{2\pi}f(0)$$

or equivalently,

$$\mathcal{F}^{-1}(\mathcal{F}f)(0) = f(0).$$

In order to prove Fourier inversion formula for any $x \in \mathbb{R}$ let us note that

$$\begin{aligned}
\widehat{f_x(y)}(\xi) = \widehat{f(x+y)}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x+y)e^{-iy\xi}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)e^{-iz\xi}e^{ix\xi}dz = e^{ix\xi}\widehat{f}(\xi).
\end{aligned}$$

Since $f_x(0) = f(x)$ then

$$\begin{aligned} f(x) &= f_x(0) = \mathcal{F}^{-1}(\mathcal{F}f_x)(0) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\mathcal{F}f_x)(\xi)d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ix\xi}\widehat{f}(\xi)d\xi = \mathcal{F}^{-1}(\mathcal{F}f)(x). \end{aligned}$$

This finishes the proof. □

Remark. As a by-product of the above proof we record the limit

$$\lim_{n\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\frac{\sin(nx)}{x}dx = f(0)$$

for any $f \in W_1^1(\mathbb{R})$.

Lemma 12.1. *If f belongs to Sobolev space $W_1^m(\mathbb{R})$ for some $m = 1, 2, \ldots$ then*

$$\widehat{f}(\xi) = o\left(\frac{1}{|\xi|^m}\right) \tag{12.4}$$

when $|\xi| \to \infty$.

Proof. Since $f \in W_1^m(\mathbb{R})$ then $f', f'', \ldots, f^{(m-1)} \in W_1^1(\mathbb{R})$. By Exercise 49 we have

$$\lim_{x\to\pm\infty} f^{(k)}(x) = 0$$

for each $k = 0, 1, \ldots, m-1$. This fact and repeated integration by parts give us

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx &= \left.\frac{e^{-ix\xi}}{-i\xi}f(x)\right|_{-\infty}^{\infty} + \frac{1}{i\xi}\int_{-\infty}^{\infty}f'(x)e^{-ix\xi}dx = \frac{1}{i\xi}\int_{-\infty}^{\infty}f'(x)e^{-ix\xi}dx \\ &= -\left.\frac{e^{-ix\xi}}{(i\xi)^2}f'(x)\right|_{-\infty}^{\infty} + \frac{1}{(i\xi)^2}\int_{-\infty}^{\infty}f''(x)e^{-ix\xi}dx \\ &= \frac{1}{(i\xi)^2}\int_{-\infty}^{\infty}f''(x)e^{-ix\xi}dx = \cdots = \frac{1}{(i\xi)^m}\int_{-\infty}^{\infty}f^{(m)}(x)e^{-ix\xi}dx \\ &= o\left(\frac{1}{|\xi|^m}\right) \end{aligned}$$

due to Riemann-Lebesgue lemma. □

The equality (12.4) allows us to consider (with respect to accuracy of calculations) the Fourier transform only from the interval $(-R, R)$ with $R > 0$ large enough i.e. we may neglect the values of $\mathcal{F}f(\xi)$ for $|\xi| > R$. This simplification justifies the following approximation of the inverse Fourier transform:

$$f^*(x) := \frac{1}{\sqrt{2\pi}}\int_{-R}^{R}\mathcal{F}f(\xi)e^{ix\xi}d\xi. \tag{12.5}$$

At the same time and without loss of generality we may assume that the function $f(x)$ has compact support. In that case it can be proved that $\mathcal{F}f(\xi)$ is smooth function for which (12.4) holds.

Definition 12.1. We say that $f \in \overset{\circ}{W}{}_1^m(-R,R)$ if $f \in W_1^m(\mathbb{R})$ and $f \equiv 0$ for $x \notin (-R,R)$.

Theorem 23. *Suppose that $f \in \overset{\circ}{W}{}_1^m(-R,R)$ is supported in a fixed interval $[a,b] \subset (-R,R)$ with $R > 0$ large enough and some $m = 2,3,\ldots$. Then*

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{1}{N^{m/(m+2)}} \sum_{n=-N/2}^{N/2-1} \mathcal{F}f\left(\frac{2n+1}{N^{m/(m+2)}}\right) e^{i\frac{x(2n+1)}{N^{m/(m+2)}}} + \mathcal{O}\left(\frac{1}{N^{(2m-2)/(m+2)}}\right) \tag{12.6}$$

uniformly in $x \in (-R,R)$ for $R = N^{2/(m+2)}$ and even N.

Proof. Since $f(x) = 0$ for $x \notin (-R,R)$ then using Fourier inversion formula (see Theorem 22) we have

$$f(x) - f^*(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}f(\xi) e^{ix\xi} d\xi - \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \mathcal{F}f(\xi) e^{ix\xi} d\xi = \frac{1}{\sqrt{2\pi}} \int_{|\xi|>R} \mathcal{F}f(\xi) e^{ix\xi} d\xi.$$

Lemma 12.1 implies then that

$$f(x) - f^*(x) = o\left(\frac{1}{R^{m-1}}\right). \tag{12.7}$$

We divide the interval $[-R,R]$ into $N+1$ subintervals $[\xi_n, \xi_{n+1}]$, $n = -N/2, \ldots, N/2-1$ such that

$$-R = \xi_{-N/2} < \xi_{-N/2+1} < \cdots < \xi_{N/2} = R,$$

where

$$\xi_n = \frac{2Rn}{N}, \quad \xi_{n+1} - \xi_n = \frac{2R}{N}.$$

Set also

$$\xi_n^* = \frac{\xi_n + \xi_{n+1}}{2} = \frac{R}{N}(2n+1).$$

Then we obtain

$$\begin{aligned} f^*(x) &= \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \mathcal{F}f(\xi) e^{ix\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=-N/2}^{N/2-1} \mathcal{F}f(\xi_n^*) e^{ix\xi_n^*} \frac{2R}{N} \\ &+ \frac{1}{\sqrt{2\pi}} \sum_{n=-N/2}^{N/2-1} \int_{\xi_n}^{\xi_{n+1}} \left(\mathcal{F}f(\xi) e^{ix\xi} - \mathcal{F}f(\xi_n^*) e^{ix\xi_n^*}\right) d\xi \\ &= \sqrt{\frac{2}{\pi}} \frac{R}{N} \sum_{n=-N/2}^{N/2-1} \mathcal{F}f(R(2n+1)/N) e^{ixR(2n+1)/N} + \mathcal{O}\left(\frac{R^3}{N^2}\right). \end{aligned} \tag{12.8}$$

Exercise 50. Prove that

$$\frac{1}{\sqrt{2\pi}}\sum_{n=-N/2}^{N/2-1}\int_{\xi_n}^{\xi_{n+1}}\left(\mathcal{F}f(\xi)e^{ix\xi}-\mathcal{F}f(\xi_n^*)e^{ix\xi_n^*}\right)d\xi=\begin{cases}\mathcal{O}\left(\frac{R^3}{N^2}\right), & \operatorname{supp} f=[a,b]\\ \mathcal{O}\left(\frac{R^5}{N^2}\right), & \operatorname{supp} f\subset(-R,R)\end{cases}$$

uniformly in $x\in[-R,R]$.

Hint. Use the Taylor expansion for the smooth function $\mathcal{F}f(\xi)e^{ix\xi}$ at the point ξ_n^*.

If we combine (12.8) with (12.7) and choose $R=N^{2/(m+2)}$ then we obtain (12.6). This finishes the proof. □

Remark. The main part of (12.6) represents some kind of DFT. In order to reconstruct f at any point $x\in[-R,R]$ we need to know only the Fourier transform of this unknown function at the points

$$\frac{2n+1}{N^{m/(m+2)}},\quad n=-N/2,\dots,N/2-1,$$

where m is the smoothness index of f. What is more, the formula (12.6) shows us that it is effective if

$$\frac{2m-2}{m+2}>\frac{m}{m+2}$$

or $m>2$. It means that f must belong to Sobolev space $\overset{\circ}{W}{}_1^m(-R,R)$ with some $m\geq 3$.

13 Some applications of discrete Fourier transform.

At first we prove Poisson summation formula.

Definition 13.1. Let f be a function such that

$$\lim_{N\to\infty}\sum_{|n|\le N} f(x+2\pi n)$$

exists pointwise in $x\in\mathbb{R}$. Then

$$f_p(x) := \sum_{n=-\infty}^{\infty} f(x+2\pi n) \tag{13.1}$$

is called the *periodization* of f.

Remark. It is clear that $f_p(x)$ is periodic with period 2π. Hence we will consider it only on the interval $[-\pi,\pi]$.

Theorem 24 (Poisson summation formula)**.** *Suppose that $f\in L^1(\mathbb{R})$. Then $f_p(x)$ from (13.1) is finite almost everywhere, satisfies $f_p(x+2\pi)=f_p(x)$ almost everywhere and is integrable on the interval $[-\pi,\pi]$. The Fourier coefficients of $f_p(x)$ are given by*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_p(x)e^{-imx}dx = \frac{1}{\sqrt{2\pi}}\mathcal{F}f(m). \tag{13.2}$$

If in addition

$$\sum_{m=-\infty}^{\infty} |\mathcal{F}f(m)| < \infty$$

then

$$\sum_{n=-\infty}^{\infty} f(x+2\pi n) = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}\mathcal{F}f(m)e^{imx}. \tag{13.3}$$

In particular, $f_p(x)$ is continuous and we have the Poisson identity

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}\mathcal{F}f(m). \tag{13.4}$$

Proof. Since $f\in L^1(\mathbb{R})$ then

$$\begin{aligned}\int_{-\pi}^{\pi}|f_p(x)|dx &\le \int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}|f(x+2\pi n)|dx = \sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi}|f(x+2\pi n)|dx\\ &= \sum_{n=-\infty}^{\infty}\int_{-\pi+2\pi n}^{\pi+2\pi n}|f(t)|dt = \int_{-\infty}^{\infty}|f(t)|dt < \infty.\end{aligned}$$

This shows that f_p is finite almost everywhere and integrable on $[-\pi,\pi]$. Applying the same calculation to $f_p(x)e^{-imx}$ allows us to integrate term by term to obtain

$$\begin{aligned}
c_m(f_p) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_p(x)e^{-imx}dx = \sum_{n=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+2\pi n)e^{-imx}dx \\
&= \sum_{n=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi+2\pi n}^{\pi+2\pi n} f(t)e^{-im(t-2\pi n)}dt = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\pi+2\pi n}^{\pi+2\pi n} f(t)e^{-imt}dt \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-imt}dt = \frac{1}{\sqrt{2\pi}}\mathcal{F}f(m).
\end{aligned}$$

Now, if the series

$$\sum_{m=-\infty}^{\infty} |\mathcal{F}f(m)|$$

converges then it is equivalent to the fact that

$$\sum_{m=-\infty}^{\infty} |c_m(f_p)| < \infty.$$

Thus, $f_p(x)$ can be represented by its Fourier series at least almost everywhere (and we can redefine $f_p(x)$ so that this representation holds pointwise) i.e.

$$f_p(x) = \sum_{m=-\infty}^{\infty} c_m(f_p)e^{imx}.$$

It means that (see (13.1))

$$\sum_{n=-\infty}^{\infty} f(x+2\pi n) = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}\mathcal{F}f(m)e^{imx}.$$

Finally, set $x=0$ to obtain the Poisson identity (13.4). □

Example. If

$$f(x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, \quad x\in\mathbb{R},$$

where $t>0$ is a parameter then it is very well-known that

$$\mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}}e^{-t\xi^2}.$$

Formula (13.3) transforms in this case to

$$\frac{1}{\sqrt{4\pi t}}\sum_{n=-\infty}^{\infty} e^{-\frac{(x+2\pi n)^2}{4t}} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-tm^2}e^{imx}$$

and the Poisson identity transforms to

$$\sqrt{\frac{\pi}{t}} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/t} = \sum_{m=-\infty}^{\infty} e^{-tm^2}.$$

As an application of the Poisson summation formula we consider the problem of reconstructing a band-limited signal from its values on the integers.

Definition 13.2. A signal $f(t)$ is called *band-limited* if it has a representation

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-2\pi\lambda}^{2\pi\lambda} F(\xi) e^{it\xi} d\xi, \tag{13.5}$$

where λ is a positive parameter and F is some integrable function.

Remark. If we set $F(\xi) = 0$ for $|\xi| > 2\pi\lambda$ then (13.5) is the inverse Fourier transform of $F \in L^1(\mathbb{R})$. In that case f is bounded and continuous.

Theorem 25 (Whittaker, Shannon, Boas)**.** *Suppose that $F \in L^1(\mathbb{R})$ and $F(\xi) = 0$ for $|\xi| > 2\pi\lambda$. If $\lambda \leq 1/2$ then for any $t \in \mathbb{R}$ we have*

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}, \tag{13.6}$$

where the fraction is equal to 1 when $t = n$. If $\lambda > 1/2$ we have

$$\left| f(t) - \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} \right| \leq \frac{1}{\sqrt{2\pi}} \left(\frac{2}{\pi} + 1 \right) \int_{\pi < |\xi| < 2\pi\lambda} |F(\xi)| d\xi. \tag{13.7}$$

Proof. Let

$$F_p(\xi) = \sum_{n=-\infty}^{\infty} F(\xi + 2\pi n)$$

be the periodization of F. The formula (13.5) shows that

$$\mathcal{F}(F)(t) = f(-t),$$

where $\mathcal{F}(F)$ denotes the Fourier transform of $F(\xi)$. Hence, by the Poisson summation formula, we have (see (13.3))

$$\sum_{n=-\infty}^{\infty} F(\xi + 2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} f(-m) e^{im\xi} = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} f(m) e^{-im\xi}.$$

Since any trigonometric Fourier series can be integrated term by term we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} F_p(\xi)e^{it\xi}d\xi &= \int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} F(\xi+2\pi n)e^{it\xi}d\xi = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} f(m)\int_{-\pi}^{\pi} e^{-i(m-t)\xi}d\xi \\ &= \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} f(m)\left.\frac{e^{i(t-m)\xi}}{i(t-m)}\right|_{-\pi}^{\pi}, \quad m\neq t \\ &= \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} f(m)\frac{e^{i(t-m)\pi}-e^{-i(t-m)\pi}}{i(t-m)} \\ &= \frac{2\pi}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty} f(m)\frac{\sin\pi(t-m)}{\pi(t-m)}.\end{aligned}$$

Now, if $\lambda \leq 1/2$ then $F(\xi)$ for $|\xi| \leq \pi$ is equal to its periodization $F_p(\xi)$ (see Definition 13.1) and

$$\int_{-\pi}^{\pi} F_p(\xi)e^{it\xi}d\xi = \sqrt{2\pi}f(t).$$

These equalities imply immediately that

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(t-n)}{\pi(t-n)}$$

so that (13.6) is proved. If $\lambda > 1/2$ then we cannot expect that $F(\xi) = F_p(\xi)$ for $|\xi| > \pi$ but using Definition 13.1 we have

$$\begin{aligned}f(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} F(\xi)e^{it\xi}d\xi + \frac{1}{\sqrt{2\pi}}\int_{\pi<|\xi|<2\pi\lambda} F(\xi)e^{it\xi}d\xi \\ &= \sum_{n=-\infty}^{\infty}\widetilde{f}(n)\frac{\sin\pi(t-n)}{\pi(t-n)} + \frac{1}{\sqrt{2\pi}}\int_{\pi<|\xi|<2\pi\lambda} F(\xi)e^{it\xi}d\xi,\end{aligned}$$

where

$$\widetilde{f}(n) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} F(\xi)e^{in\xi}d\xi.$$

Therefore,

$$\begin{aligned}f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(t-n)}{\pi(t-n)} \\ + \sum_{n=-\infty}^{\infty}\left(\widetilde{f}(n)-f(n)\right)\frac{\sin\pi(t-n)}{\pi(t-n)} + \frac{1}{\sqrt{2\pi}}\int_{\pi<|\xi|<2\pi\lambda} F(\xi)e^{it\xi}d\xi.\end{aligned}$$

Here the middle series is equal to

$$-\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\left(\int_{\pi<|\xi|<2\pi\lambda} F(\xi)e^{in\xi}d\xi\right)\frac{\sin\pi(t-n)}{\pi(t-n)}$$

or

$$-\frac{1}{\sqrt{2\pi}}\int_{\pi<|\xi|<2\pi\lambda} F(\xi)\left(\sum_{n=-\infty}^{\infty}\frac{\sin\pi(t-n)}{\pi(t-n)}e^{in\xi}\right)d\xi.$$

Exercise 51. Prove that

$$\sum_{n=-\infty}^{\infty}\frac{\sin\pi(t-n)}{\pi(t-n)}e^{in\xi} = -\frac{2}{\pi}e^{it\xi}.$$

Hint. Show that

$$c_n\left(-\frac{2}{\pi}e^{it\xi}\right) = \frac{\sin\pi(t-n)}{\pi(t-n)}.$$

Using Exercise 51 we have

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(t-n)}{\pi(t-n)} + \left(\frac{2}{\pi\sqrt{2\pi}}+\frac{1}{\sqrt{2\pi}}\right)\int_{\pi<|\xi|<2\pi\lambda} F(\xi)e^{it\xi}d\xi.$$

Thus

$$\left|f(t) - \sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(t-n)}{\pi(t-n)}\right| \le \frac{1}{\sqrt{2\pi}}\left(\frac{2}{\pi}+1\right)\int_{\pi<|\xi|<2\pi\lambda}|F(\xi)|d\xi.$$

This proves the theorem. □

Theorem 25 shows that in order to reconstruct a band-limited signal $f(t)$ it is enough to know the values of this signal at integers $f(n)$. In turn, to evaluate $f(n)$ it is enough to use IDFT of $F(\xi)$, see (13.5). Indeed, let us assume without loss of generality that $\lambda = 1/2$. Then

$$f(n) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} F(\xi)e^{in\xi}d\xi.$$

If F is smooth enough (say $F \in C^2[-\pi,\pi]$) then formula (12.8) gives ($R = \pi$ and $N >> 1$)

$$f(n) = \frac{\sqrt{2\pi}}{N}\sum_{k=-N/2}^{N/2-1} F_k e^{i\frac{\pi n(2k+1)}{N}} + \mathcal{O}\left(\frac{1}{N^2}\right) = \frac{\sqrt{2\pi}}{N}e^{i\frac{\pi n}{N}}\sum_{k=-N/2}^{N/2-1} F_k e^{i\frac{2\pi kn}{N}} + \mathcal{O}\left(\frac{1}{N^2}\right),$$

where F_k denotes the value of $F(\xi)$ at the point $\pi(2k+1)/N$. Therefore, up to the accuracy of calculations,

$$f(n) \approx \frac{\sqrt{2\pi}}{N}e^{i\frac{\pi n}{N}}\mathcal{F}^{-1}(F_k)_n$$

i.e. for even integer N large enough

$$f(t) \approx \sum_{n=-\infty}^{\infty}\frac{\sqrt{2\pi}}{N}e^{i\frac{\pi n}{N}}\mathcal{F}^{-1}(F_k)_n\frac{\sin\pi(t-n)}{\pi(t-n)}.$$

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