Spectral theory of elliptic differential operators

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Lecture Notes

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Contents

1 Inner product spaces and Hilbert spaces

A collection of elements is called a complex (real) *vector space* (*linear space*) H if the following axioms are satisfied:

1) To every pair $x, y \in H$ there corresponds a vector $x+y$, called the sum, with the properties:

   a) $x + y = y + x$
   b) $x + (y + z) = (x + y) + z \equiv x + y + z$
   c) there exists unique $0 \in H$ such that $x + 0 = x$
   d) for every $x \in H$ there exists unique $y_1 \in H$ such that $x+y_1 = 0$. We denote $y_1 := -x$.

2) For every $x \in H$ and every $\lambda, \mu \in \mathbb{C}$ there corresponds a vector $\lambda \cdot x$ such that

   a) $\lambda(\mu x) = (\lambda\mu)x \equiv \lambda\mu x$
   b) $(\lambda + \mu)x = \lambda x + \mu x$
   c) $\lambda(x + y) = \lambda x + \lambda y$
   d) $1 \cdot x = x$.

Definition. For a linear space H every mapping $(\cdot, \cdot) : H \times H \to \mathbb{C}$ is called an *inner product* or a *scalar product* if

1) $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$
2) $(x, y + z) = (x, y) + (x, z)$
3) $(\lambda x, y) = \lambda(x, y)$
4) $(x, y) = \overline{(y, x)}$

for every $x, y, z \in H$ and $\lambda \in \mathbb{C}$. A linear space equipped with an inner product is called an *inner product space*.

An immediate consequence of this definition is that

$$\begin{aligned}(\lambda x + \mu y, z) &= \lambda(x, z) + \mu(y, z),\\ (x, \lambda y) &= \overline{\lambda}(x, y)\end{aligned}$$

for every $x, y, z \in H$ and $\lambda, \mu \in \mathbb{C}$.

Example 1.1. On the complex Euclidean space $H = \mathbb{C}^n$ the standard inner product is

$$(x, y) = \sum_{j=1}^{n} x_j \overline{y_j},$$

where $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$.

Example 1.2. On the linear space $C[a, b]$ of continuous complex-valued functions, the formula

$$(f, g) = \int_a^b f(x)\overline{g(x)}dx$$

defines an inner product.

Definition. Suppose H is an inner product space. One calls

1) $x \in H$ *orthogonal* to $y \in H$ if $(x, y) = 0$.

2) a system $\{x_\alpha\}_{\alpha \in A} \subset H$ *orthonormal* if $(x_\alpha, x_\beta) = \delta_{\alpha,\beta} = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$, where A is some index set.

3) $\|x\| := \sqrt{(x, x)}$ is called the *length* of $x \in H$.

Exercise 1. Prove the *Theorem of Pythagoras*: If $\{x_j\}_{j=1}^k, k \in \mathbb{N}$ is an orthonormal system in an inner product space H, then

$$\|x\|^2 = \sum_{j=1}^k |(x, x_j)|^2 + \left\| x - \sum_{j=1}^k (x, x_j)x_j \right\|^2$$

for every $x \in H$.

Exercise 2. Prove *Bessel's inequality*: If $\{x_j\}_{j=1}^k, k \leq \infty$ is an orthonormal system then

$$\sum_{j=1}^k |(x, x_j)|^2 \leq \|x\|^2,$$

for every $x \in H$.

Exercise 3. Prove the *Cauchy-Schwarz-Bunjakovskii inequality*:

$$|(x, y)| \leq \|x\| \|y\|, \quad x, y \in H.$$

Prove also that $(\cdot, \cdot)$ is continuous as a map from $H \times H$ to $\mathbb{C}$.

If H is an inner product space, then

$$\|x\| := \sqrt{(x, x)}$$

has the following properties:

1) $\|x\| \geq 0$ for every $x \in H$ and $\|x\| = 0$ if and only if $x = 0$.

2) $\|\lambda x\| = |\lambda| \|x\|$ for every $x \in H$ and $\lambda \in \mathbb{C}$.

3) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in H$. This is the *triangle inequality*.

The function $\|\cdot\| = \sqrt{(\cdot,\cdot)}$ is thus a *norm* on H. It is called the norm *induced by the inner product*.

Every inner product space H is a normed space under the induced norm. The neighborhood of $x \in H$ is the open ball $B_r(x) = \{y \in H : \|x - y\| < r\}$. This system of neighborhoods defines the *norm topology* on H such that:

1) The addition $x + y$ is a continuous map $H \times H \to H$.

2) The scalar multiplication $\lambda \cdot x$ is a continuous map $\mathbb{C} \times H \to H$.

Definition. 1) A sequence $\{x_j\}_{j=1}^{\infty} \subset H$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|x_k - x_j\| < \varepsilon$ for $k, j \geq n_0$.

2) A sequence $\{x_j\}_{j=1}^{\infty} \subset H$ is said to be *convergent* if there exists $x \in H$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|x - x_j\| < \varepsilon$ whenever $j \geq n_0$.

3) The inner product space H is *complete space* if every Cauchy sequence in H converges.

Corollary. *1) Every convergent sequence is a Cauchy sequence.*

2) If $\{x_j\}_{j=1}^{\infty}$ converges to $x \in H$ then

$$\lim_{j\to\infty} \|x_j\| = \|x\| .$$

Definition. (J. von Neumann, 1925) A *Hilbert space* is an inner product space which is complete (with respect to its norm topology).

Exercise 4. Prove that in an inner product space the norm induced by this inner product satisfies the *parallelogram law*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

Exercise 5. Prove that if in a normed space H the parallelogram law holds, then there is an inner product on H such that $\|x\|^2 = (x,x)$ and that this inner product is defined by the *polarization identity*

$$(x,y) := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

Exercise 6. Prove that on $C[a,b]$ the norm

$$\|f\| = \max_{x\in[a,b]} |f(x)|$$

is not induced by an inner product.

Exercise 7. Give an example of an inner product space which is not complete.

Next we list some examples of Hilbert spaces.

1) The Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$.

2) The matrix space $M_n(\mathbb{C})$ consisting of $n\times n$ -matrices whose elements are complex numbers. For $A, B \in M_n(\mathbb{C})$ the inner product is given by

$$(A, B) = \sum_{k,j=1}^{n} a_{kj}\overline{b_{kj}} = \mathrm{Tr}\,(AB^*),$$

where $B^* = \overline{B}^T$.

3) The *sequence space* $l^2(\mathbb{C})$ defined by

$$l^2(\mathbb{C}) := \left\{ \{x_j\}_{j=1}^{\infty}, x_j \in \mathbb{C} : \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}.$$

The estimates

$$|x_j + y_j|^2 \le 2\left(|x_j|^2 + |y_j|^2\right), \quad |\lambda x_j|^2 = |\lambda|^2 |x_j|^2$$

and

$$|x_j y_j| \le \frac{1}{2}\left(|x_j|^2 + |y_j|^2\right)$$

imply that $l^2(\mathbb{C})$ is a linear space. Let us define the inner product by

$$(x, y) := \sum_{j=1}^{\infty} x_j \overline{y_j}$$

and prove that $l^2(\mathbb{C})$ is complete. Suppose that $\{x^{(k)}\}_{k=1}^{\infty} \in l^2(\mathbb{C})$ is a Cauchy sequence. Then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\left\|x^{(k)} - x^{(m)}\right\|^2 = \sum_{j=1}^{\infty} |x_j^{(k)} - x_j^{(m)}|^2 < \varepsilon^2$$

for $k, m \ge n_0$. It implies that

$$|x_j^{(k)} - x_j^{(m)}| < \varepsilon, \quad j = 1, 2, \ldots$$

or that $\{x_j^{(k)}\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$ for every $j = 1, 2, \ldots$. Since $\mathbb{C}$ is a complete space then $\{x_j^{(k)}\}_{k=1}^{\infty}$ converges for every fixed $j = 1, 2, \ldots$ i.e. there exists $x_j \in \mathbb{C}$ such that

$$x_j = \lim_{k\to\infty} x_j^{(k)}.$$

This fact and

$$\sum_{j=1}^{l} |x_j^{(k)} - x_j^{(m)}|^2 < \varepsilon^2, \quad l \in \mathbb{N}$$

imply that

$$\lim_{m\to\infty}\sum_{j=1}^{l}|x_j^{(k)}-x_j^{(m)}|^2=\sum_{j=1}^{l}|x_j^{(k)}-x_j|^2\le\varepsilon^2$$

for all $k\ge n_0$ and $l\in\mathbb{N}$. Therefore the sequence

$$s_l:=\sum_{j=1}^{l}|x_j^{(k)}-x_j|^2,\quad k\ge n_0$$

is a monotone increasing sequence which is bounded from above by ε^2. Hence this sequence has a limit with the same upper bound i.e.

$$\sum_{j=1}^{\infty}|x_j^{(k)}-x_j|^2=\lim_{l\to\infty}\sum_{j=1}^{l}|x_j^{(k)}-x_j|^2\le\varepsilon^2.$$

That's why we may conclude that

$$\|x\|\le\left\|x^{(k)}\right\|+\left\|x^{(k)}-x\right\|\le\left\|x^{(k)}\right\|+\varepsilon$$

and $x\in l^2(\mathbb{C})$.

4) The *Lebesgue space* $L^2(\Omega)$, where $\Omega\subset\mathbb{R}^n$ is an open set. The space $L^2(\Omega)$ consists of all Lebesgue measurable functions f which are square integrable i.e.

$$\int_\Omega|f(x)|^2dx<\infty.$$

It is a linear space with the inner product

$$(f,g)=\int_\Omega f(x)\overline{g(x)}dx$$

and the Riesz-Fisher theorem reads as: $L^2(\Omega)$ is a Hilbert space.

5) The *Sobolev spaces* $W_2^k(\Omega)$ consisting of functions $f\in L^2(\Omega)$ whose weak or distributional derivatives $D^\alpha f$ also belong to $L^2(\Omega)$ up to order $|\alpha|\le k, k=1,2,\ldots$. On the space $W_2^k(\Omega)$ the natural inner product is

$$(f,g)=\sum_{|\alpha|\le k}\int_\Omega D^\alpha f(x)\overline{D^\alpha g(x)}dx.$$

Definition. Let H be an inner product space. For any subspace $M\subset H$ the *orthogonal complement* of M is defined as

$$M^\perp:=\{y\in H:(y,x)=0,x\in M\}.$$

Remark. It is clear that $M^\perp$ is a linear subspace of H. Moreover, $M\cap M^\perp=\{0\}$ since $0\in M$ always.

Definition. A *closed subspace* of a Hilbert space H is a linear subspace of H which is closed (i.e. $\overline{M} = M$) with respect to the induced norm.

Remark. The subspace $M^{\perp}$ is closed if M is any subset of a Hilbert space.

Theorem 1 (*Projection theorem*)**.** *Suppose M is a closed subspace of a Hilbert space H. Then every $x \in H$ has the unique representation as*

$$x = u + v,$$

where $u \in M$ and $v \in M^{\perp}$, or equivalently,

$$H = M \oplus M^{\perp}.$$

Moreover, one has that

$$\|v\| = \inf_{y \in M} \|x - y\| := d(x, M).$$

Proof. Let $x \in H$. Then

$$d := d(x, M) \equiv \inf_{y \in M} \|x - y\| \leq \|x - u\|$$

for any $u \in M$. The definition of infimum implies that there exists a sequence $\{u_j\}_{j=1}^{\infty} \subset M$ such that

$$d = \lim_{j \to \infty} \|x - u_j\|.$$

The parallelogram law implies that

$$\begin{aligned} \|u_j - u_k\|^2 &= \|(u_j - x) + (x - u_k)\|^2 \\ &= 2\|u_j - x\|^2 + 2\|x - u_k\|^2 - 4\left\|x - \frac{u_j + u_k}{2}\right\|^2. \end{aligned}$$

Since $(u_j + u_k)/2 \in M$ then

$$\|u_j - u_k\|^2 \leq 2\|u_j - x\|^2 + 2\|x - u_k\|^2 - 4d^2 \to 2d^2 + 2d^2 - 4d^2 = 0$$

as $j, k \to \infty$. Hence $\{u_j\}_{j=1}^{\infty} \subset M$ is a Cauchy sequence in the Hilbert space H. It means that there exists $u \in H$ such that

$$u = \lim_{j \to \infty} u_j.$$

But $M = \overline{M}$ implies that $u \in M$. By construction one has that

$$d = \lim_{j \to \infty} \|x - u_j\| = \|x - u\|.$$

Let us denote $v := x - u$ and show that $v \in M^{\perp}$. For any $y \in M, y \neq 0$ introduce the number

$$\alpha = -\frac{(v, y)}{\|y\|^2}.$$

Since $u - \alpha y \in M$ we have

$$\begin{aligned} d^2 &\leq \|x-(u-\alpha y)\|^2 = \|v+\alpha y\|^2 = \|v\|^2 + (v,\alpha y) + (\alpha y, v) + |\alpha|^2 \|y\|^2 \\ &= d^2 - \frac{\overline{(v,y)}(v,y)}{\|y\|^2} - \frac{(v,y)(y,v)}{\|y\|^2} + \frac{|(v,y)|^2}{\|y\|^2} = d^2 - \frac{|(v,y)|^2}{\|y\|^2}. \end{aligned}$$

This inequality implies that $(y, v) = 0$. It means that $v \in M^{\perp}$. In order to prove uniqueness assume that $x = u_1 + v_1 = u_2 + v_2$, where $u_1, u_2 \in M$ and $v_1, v_2 \in M^{\perp}$. It follows that

$$u_1 - u_2 = v_2 - v_1 \in M \cap M^{\perp}.$$

But $M \cap M^{\perp} = \{0\}$ so that $u_1 = u_2$ and $v_1 = v_2$. □

Corollary 1 (*Riesz-Frechet theorem*)**.** *If T is a linear continuous functional on the Hilbert space H then there exists a unique $h \in H$ such that $T(x) = (x, h)$ for all $x \in H$. Moreover, $\|T\|_{H\to\mathbb{C}} = \|h\|$.*

Proof. If $T \equiv 0$ then $h = 0$ will do. If $T \neq 0$ then there exists $v_0 \in H$ such that $T(v_0) \neq 0$. Let

$$M := \{u \in H : T(u) = 0\}.$$

Since T is linear and continuous then M is a closed subspace. It follows from Thereom 1 that

$$H = M \oplus M^{\perp}$$

i.e. every $x \in H$ has the unique representation as $x = \widetilde{u} + \widetilde{v}$. Since $v_0 \neq 0$ then $v_0 \in M^{\perp}$. Therefore, for every $x \in H$, we can define

$$u := x - \frac{T(x)}{T(v_0)} v_0.$$

Then $T(u) = 0$ i.e. $u \in M$. It follows that

$$(x, v_0) = (u, v_0) + \frac{T(x)}{T(v_0)} \|v_0\|^2 = \frac{T(x)}{T(v_0)} \|v_0\|^2$$

or

$$T(x) = \frac{T(v_0)}{\|v_0\|^2}(x, v_0) = \left(x, \frac{\overline{T(v_0)}}{\|v_0\|^2} v_0\right),$$

which is of the desired form. The uniqueness of h can be seen as follows. If $T(x) = (x, h) = (x, \widetilde{h})$ then $(x, h - \widetilde{h}) = 0$ for all $x \in H$. In particular $\left\|h - \widetilde{h}\right\|^2 = (h - \widetilde{h}, h - \widetilde{h}) = 0$ i.e. $h = \widetilde{h}$. It remains to prove the statement about the norm $\|T\|_{H\to\mathbb{C}} = \|T\|$. Firstly,

$$\|T\| = \sup_{\|x\|\leq 1} |T(x)| = \sup_{\|x\|\leq 1} |(x, h)| \leq \|h\|.$$

On the other hand $T(h/\|h\|) = \|h\|$ implies that $\|T\| \geq \|h\|$. Thus $\|T\| = \|h\|$. This finishes the proof. □

Corollary 2. *If M is a linear subspace of a Hilbert space H then*

$$M^{\perp\perp} := \left(M^{\perp}\right)^{\perp} = \overline{M}.$$

Proof. It is not so difficult to check that

$$M^{\perp} = \left(\overline{M}\right)^{\perp}.$$

That's why

$$M^{\perp\perp} = \left(\left(\overline{M}\right)^{\perp}\right)^{\perp}$$

and Theorem 1 implies that

$$H = \overline{M} \oplus \left(\overline{M}\right)^{\perp}, \quad H = \left(\overline{M}\right)^{\perp} \oplus M^{\perp\perp}.$$

Uniqueness of this representation guarantees that $M^{\perp\perp} = \overline{M}$. □

Remark. In the frame of this theorem we have that

$$\|x\|^2 = \|u\|^2 + \|v\|^2, \quad \|v\|^2 = (x, v).$$

Definition. Let $A \subset H$ be a subset of an inner product space. The subset

$$\operatorname{span} A := \left\{ x \in H : x = \sum_{j=1}^{k} \lambda_j x_j, x_j \in A, \lambda_j \in \mathbb{C} \right\}$$

is called the *linear span* of A.

Definition. Let H be a Hilbert space.

1) A subset $B \subset H$ is called a *basis* of H if B is linearly independent in H and

$$\overline{\operatorname{span} B} = H$$

i.e. for every $x \in H$ and every $\varepsilon > 0$ there exist $k \in \mathbb{N}$ and $\{c_j\}_{j=1}^{k} \subset \mathbb{C}$ such that

$$\left\| x - \sum_{j=1}^{k} c_j x_j \right\| < \varepsilon, \quad x_j \in B.$$

2) The Hilbert space is called *separable* if it has a countable or finite basis.

3) An orthonormal system $B = \{x_\alpha\}_{\alpha \in A}$ in H which is a basis is called an *orthonormal basis*.

By the Gram-Schmidt orthonormalization we may conclude that every separable Hilbert space has an orthonormal basis.

Theorem 2 (Characterization of an orthonormal basis)**.** *Let $B = \{x_j\}_{j=1}^{\infty}$ be an orthonormal system in a separable Hilbert space H. Then the following statements are equivalent:*

1) *B is maximal i.e. it is not a proper subset of any other orthonormal system.*

2) *For every $x \in H$ the condition $(x, x_j) = 0, j = 1, 2, \dots$ implies that $x = 0$.*

3) *Every $x \in H$ has the* Fourier expansion

$$x = \sum_{j=1}^{\infty} (x, x_j) x_j$$

i.e.

$$\left\| x - \sum_{j=1}^{k} (x, x_j) x_j \right\| \to 0, \quad k \to \infty.$$

This means that B is an orthonormal basis.

4) *Every pair $x, y \in H$ satisfies the* completeness relation

$$(x, y) = \sum_{j=1}^{\infty} (x, x_j) \overline{(y, x_j)}.$$

5) *Every $x \in H$ satisfies the* Parseval equality

$$\|x\|^2 = \sum_{j=1}^{\infty} |(x, x_j)|^2.$$

Proof. **1)$\Rightarrow$ 2)** Suppose that there is $z \in H, z \neq 0$ such that $(z, x_j) = 0$ for all $j = 1, 2, \dots$. Then

$$B' := \left\{ \frac{z}{\|z\|}, x_1, x_2, \dots \right\}$$

is an orthonormal system in H. This fact implies that B is not maximal. It contradicts 1) and proves 2).

2)$\Rightarrow$3) Given $x \in H$ introduce the sequence

$$x^{(k)} = \sum_{j=1}^{k} (x, x_j) x_j.$$

Theorem of Pythagoras and Bessel's inequality (Exercises 1 and 2) imply that

$$\left\| x^{(k)} \right\|^2 = \sum_{j=1}^{k} |(x, x_j)|^2 \le \|x\|^2.$$

It follows that

$$\sum_{j=1}^{\infty} |(x, x_j)|^2$$

converges. That's why, for $m < k$,

$$\left\| x^{(k)} - x^{(m)} \right\|^2 = \sum_{j=m+1}^{k} |(x, x_j)|^2 \to 0$$

as $k, m \to \infty$. Hence $x^{(k)}$ is a Cauchy sequence in H. Thus there exists $y \in H$ such that

$$y = \lim_{k\to\infty} x^{(k)} = \sum_{j=1}^{\infty} (x, x_j) x_j.$$

Next, since the inner product is continuous we deduce that

$$(y, x_j) = \lim_{k\to\infty} (x^{(k)}, x_j) = (x, x_j)$$

for any $j = 1, 2, \ldots$. Therefore $(y - x, x_j) = 0$ for any $j = 1, 2, \ldots$. Part 2) implies that $y = x$ and part 3) follows.

3)⇒4) Let $x, y \in H$. We know from part 3) that

$$x = \sum_{j=1}^{\infty} (x, x_j) x_j, \quad y = \sum_{k=1}^{\infty} (y, x_k) x_k.$$

Continuity of the inner product and orthonormality of $\{x_j\}_{j=1}^{\infty}$ allow us to conclude that

$$(x, y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (x, x_j) \overline{(y, x_k)} (x_j, x_k) = \sum_{j=1}^{\infty} (x, x_j) \overline{(y, x_j)}.$$

4)⇒5) Take $y = x$ in part 4).

5)⇒1) Suppose that B is not maximal. Then we can add a unit vector $z \in H$ to it which is orthogonal to B. Parseval's equality gives then

$$1 = \|z\|^2 = \sum_{j=1}^{\infty} |(z, x_j)|^2 = 0.$$

This contradiction proves the result.

□

Exercise 8. Let $\{x_j\}_{j=1}^{\infty}$ be an orthonormal system in an inner product space H. Let $x \in H$, $\{c_j\}_{j=1}^{k} \subset \mathbb{C}$ and $k \in \mathbb{N}$. Prove that

$$\left\| x - \sum_{j=1}^{k} (x, x_j) x_j \right\| \le \left\| x - \sum_{j=1}^{k} c_j x_j \right\|.$$

2 Symmetric operators in the Hilbert space

Assume that H is a Hilbert space. A *linear operator* from H to H is a mapping

$$A : D(A) \subset H \to H,$$

where $D(A)$ is a linear subspace of H and A satisfies the condition

$$A(\lambda x + \mu y) = \lambda Ax + \mu Ay$$

for all $\lambda, \mu \in \mathbb{C}$ and $x, y \in D(A)$. The space $D(A)$ is called the *domain* of A. The space

$$N(A) := \{x \in D(A) : Ax = 0\}$$

is called the *nullspace* (or the *kernel*) of A. The space

$$R(A) := \{y \in H : y = Ax \, \text{for some} \, x \in D(A)\}$$

is called the *range* of A. Both $N(A)$ and $R(A)$ are linear subspaces of H. We say that A is *bounded* if there exists $M > 0$ such that

$$\|Ax\| \le M \|x\|, \quad x \in D(A).$$

We say that A is *densely defined* if $\overline{D(A)} = H$. In such case A can be extended to A_{ex} which will be defined on the whole H with the same norm estimate and we may define

$$\|A\|_{H \to H} := \inf\{M : \|Ax\| \le M \|x\|, x \in D(A)\}$$

or equivalently

$$\|A\|_{H \to H} = \sup_{\|x\|=1} \|Ax\|.$$

Exercise 9 (Hellinger-Toeplitz)**.** Suppose that $D(A) = H$ and

$$(Ax, y) = (x, Ay), \quad x, y \in H.$$

Prove that A is bounded.

Example 2.1 (Integral operator in L^2)**.** Suppose that $K(s,t) \in L^2((a,b) \times (a,b))$. Define the integral operator $\widehat{K}$ as

$$\widehat{K} f(s) = \int_a^b K(s,t) f(t) dt, \quad f \in L^2(a,b).$$

Let us prove that $\widehat{K}$ is bounded. Indeed,

$$\begin{aligned}
\left\|\widehat{K} f\right\|_{L^2(a,b)}^2 &= \int_a^b |\widehat{K} f(s)|^2 ds = \int_a^b \left| \int_a^b K(s,t) f(t) dt \right|^2 ds \\
&= \int_a^b \left| (K(s,\cdot), \overline{f})_{L^2} \right|^2 ds \le \int_a^b \|K(s,\cdot)\|_{L^2}^2 \left\|\overline{f}\right\|_{L^2}^2 ds \\
&= \int_a^b \left(\int_a^b |K(s,t)|^2 dt \int_a^b |f(t)|^2 dt \right) ds \\
&= \|K\|_{L^2((a,b)\times(a,b))}^2 \|f\|_{L^2(a,b)}^2
\end{aligned}$$

where we have made use of Cauchy-Schwarz-Bunjakovskii inequality. That's why we have

$$\left\|\widehat{K}\right\|_{L^2\to L^2} \leq \|K\|_{L^2((a,b)\times(a,b))}.$$

The norm

$$\|K\|_{L^2((a,b)\times(a,b))} := \left\|\widehat{K}\right\|_{HS}$$

is called the *Hilbert-Schmidt norm* of $\widehat{K}$.

Example 2.2 (Differential operator in L^2)**.** Consider the differential operator

$$A := i\frac{d}{dt}$$

in $L^2(0,1)$ with the domain

$$D(A) = \left\{f \in C^1[0,1] : f(0) = f(1) = 0\right\}.$$

First of all we have that $\overline{D(A)} = L^2$. Moreover, integration by parts gives

$$(Af, g) = \int_0^1 if'(t)\overline{g(t)}dt = i\left[f\overline{g}|_0^1 - \int_0^1 f(t)\overline{g'(t)}dt\right] = \int_0^1 f(t)\overline{ig'(t)}dt = (f, Ag)$$

for all $f, g \in D(A)$. Let us now consider the sequence

$$u_n(t) := \sin(n\pi t), \quad n = 1, 2, \ldots.$$

Clearly $u_n \in D(A)$ and

$$\|u_n\|_{L^2}^2 = \int_0^1 |\sin(n\pi t)|^2 dt = \frac{1}{2}.$$

But

$$\|Au_n\|_{L^2}^2 = \int_0^1 \left|i\frac{d}{dt}\sin(n\pi t)\right|^2 dt = (n\pi)^2 \int_0^1 |\cos(n\pi t)|^2 dt = (n\pi)^2 \frac{1}{2} = (n\pi)^2 \|u_n\|_{L^2}^2.$$

Therefore A is not bounded. This shows that $D(A) = H$ is an essential assumption in Exercise 9.

We assume later on that $\overline{D(A)} = H$ i.e. that A is densely defined in any case.

Definition. The *graph* $\Gamma(A)$ of a linear operator A in the Hilbert space H is defined as

$$\Gamma(A) := \{(x; y) \in H \times H : x \in D(A) \,\text{and}\, y = Ax\}.$$

Remark. The graph $\Gamma(A)$ is a linear subspace of the Hilbert space $H \times H$. The inner product in $H \times H$ can be defined as

$$((x_1; y_1), (x_2; y_2))_{H\times H} := (x_1, x_2)_H + (y_1, y_2)_H$$

for any $(x_1; y_1), (x_2; y_2) \in H \times H$.

Definition. The operator A is called *closed* if $\Gamma(A) = \overline{\Gamma(A)}$. We denote this fact by $A = \overline{A}$.

By definition, the *criterion for closedness* is that

$$\begin{cases} x_n \in D(A) \\ x_n \to x \\ Ax_n \to y \end{cases} \quad \Rightarrow \quad \begin{cases} x \in D(A) \\ y = Ax. \end{cases}$$

The reader is asked to verify that it is also possible to use seemingly weaker, but equivalent, criterion:

$$\begin{cases} x_n \in D(A) \\ x_n \xrightarrow{w} x \\ Ax_n \xrightarrow{w} y \end{cases} \quad \Rightarrow \quad \begin{cases} x \in D(A) \\ y = Ax, \end{cases}$$

where $x_n \xrightarrow{w} x$ indicates weak convergence in the sense that

$$(x_n, y) \to (x, y)$$

for all $y \in H$.

Definition. Let A and A_1 be two linear operators in a Hilbert space H. We say that A_1 is an *extension* of A (and A is a *restriction* of A_1) if $D(A) \subset D(A_1)$ and $Ax = A_1x$ for all $x \in D(A)$. We denote this fact by $A \subset A_1$ and $A = A_1|_{D(A)}$.

Definition. We say that A is *closable* if A has an extension A_1 and $A_1 = \overline{A_1}$. The *closure* of A, denoted by $\overline{A}$, is the smallest closed extension of A if it exists, i.e.

$$\overline{A} = \bigcap_{\substack{A \subset A_1 \\ A_1 = \overline{A_1}}} A_1.$$

Here, by $A_1 \cap \widetilde{A_1}$, we mean the operator whose domain is $D(A_1 \cap \widetilde{A_1}) := D(A_1) \cap D(\widetilde{A_1})$ and

$$(A_1 \cap \widetilde{A_1})x := A_1x = \widetilde{A_1}x, \quad x \in D(A_1 \cap \widetilde{A_1}),$$

whenever $A \subset A_1 = \overline{A_1}$ and $A \subset \widetilde{A_1} = \overline{\widetilde{A_1}}$.

If A is closable then $\Gamma\left(\overline{A}\right) = \overline{\Gamma(A)}$.

Definition. Consider the subspace

$$D^* := \{v \in H : \text{there exists}\, h \in H \,\text{such that}\, (Ax, v) = (x, h) \,\text{for all}\, x \in D(A)\}.$$

The operator A^* with the domain $D(A^*) := D^*$ and mapping $A^*v = h$ is called the *adjoint operator* of A.

Exercise 10. Prove that A^* exists as unique linear operator.

Remark. The adjoint operator is maximal among all linear operators B (in the sense that $B \subset A^*$) which satisfy

$$(Ax, y) = (x, By)$$

for all $x \in D(A)$ and $y \in D(B)$.

Example 2.3. Consider the operator

$$Af(x) := x^{-\alpha} f(x), \quad \alpha > 1/2$$

in the Hilbert space $H = L^2(0,1)$. Let us define

$$D(A) := \left\{ f \in L^2(0,1) : f(x) = \chi_n(x) g(x), g \in L^2 \text{ for some } n \in \mathbb{N} \right\},$$

where

$$\chi_n(x) = \begin{cases} 0, & 0 \le x \le 1/n \\ 1, & 1/n < x \le 1. \end{cases}$$

It is clear that $\overline{D(A)} = L^2(0,1)$. For $v \in D(A^*)$ we have

$$(Af, v) = \int_0^1 x^{-\alpha} \chi_n(x) g(x) \overline{v(x)} dx = \int_0^1 f(x) \overline{x^{-\alpha} v(x)} dx = (f, A^* v).$$

That's why we may conclude that

$$D(A^*) = \left\{ v \in L^2 : x^{-\alpha} v \in L^2 \right\}.$$

Let us show that A is not closed. To see this take the sequence

$$f_n(x) = \begin{cases} x^\alpha, & 1/n < x \le 1 \\ 0, & 0 \le x \le 1/n. \end{cases}$$

Then $f_n \in D(A)$ and

$$Af_n(x) = \begin{cases} 1, & 1/n < x \le 1 \\ 0, & 0 \le x \le 1/n. \end{cases}$$

If we assume that $A = \overline{A}$ then

$$\begin{cases} f_n \in D(A) \\ f_n \to x^\alpha \\ Af_n \to 1 \end{cases} \Rightarrow \begin{cases} x^\alpha \in D(A) \\ 1 = Ax^\alpha. \end{cases}$$

But $x^\alpha \notin D(A)$. This contradiction shows us that A is not closed. It is not bounded either since $\alpha > 1/2$.

Theorem 1. *Let A be linear and densely defined operator. Then*

1) $A^* = \overline{A^*}$.

2) A is closable if and only if $\overline{D(A^*)} = H$. In this case $A^{**} := (A^*)^* = \overline{A}$.

3) If A is closable then $\left(\overline{A}\right)^* = A^*$.

Proof. 1) Let us define in $H \times H$ the linear and bounded operator V as the mapping

$$V : (u; v) \to (v; -u).$$

It has the property $V^2 = -I$. The equality $(Au, v) = (u, A^*v)$ for $u \in D(A)$ and $v \in D(A^*)$ can be rewritten as

$$(V(u; Au), (v; A^*v))_{H\times H} = 0.$$

It implies that $\Gamma(A^*) \perp V\Gamma(A)$ and $\Gamma(A^*) \perp \overline{V\Gamma(A)}$. It means (see Theorem 1 in Section 1) that $\Gamma(A^*) = \left(\overline{V\Gamma(A)}\right)^{\perp}$. Since the orthogonal complement is always closed then $\Gamma(A^*)$ is closed. This proves 1).

2) Assume $\overline{D(A^*)} = H$. Then we can define $A^{**} := (A^*)^*$ and due to part 1) we may conclude that

$$\Gamma(A^{**}) \perp \overline{V\Gamma(A^*)}.$$

Next, since $\overline{\Gamma(A)}$ is a closed subspace of $H \times H$ we have $\overline{\Gamma(A)} = \left(\overline{\Gamma(A)}\right)^{\perp\perp}$. Since $V^2 = -I$ and V is bounded then

$$\overline{\Gamma(A)} = -\left(V^2\overline{\Gamma(A)}\right)^{\perp\perp} = -\left(V\left(\overline{V\Gamma(A)}\right)^{\perp}\right)^{\perp} = -(V\Gamma(A^*))^{\perp}$$

by 1). Hence

$$\overline{\Gamma(A)} \perp \overline{V\Gamma(A^*)}.$$

It follows that

$$\overline{\Gamma(A)} = \Gamma(A^{**})$$

or

$$\Gamma(\overline{A}) = \Gamma(A^{**})$$

or

$$\overline{A} = A^{**}.$$

This proves 2) in one direction. Let us assume now that A is closable but $\overline{D(A^*)} \neq H$. It is equivalent to the fact that there exists $u_0 \neq 0$ such that $u_0 \perp \overline{D(A^*)}$. In that case for any $v \in D(A^*)$ the element $(u_0; 0) \in H \times H$ is orthogonal to $(v; A^*v) \in \Gamma(A^*) \subset H \times H$. This is equivalent to (see 1)) $(u_0; 0) \in \overline{V\Gamma(A)}$. Since A is closable and V is bounded then $(u_0; 0) \in V\Gamma(\overline{A})$ or

$$-V(u_0; 0) = (0; u_0) \in \Gamma(\overline{A})$$

or $\overline{A}(0) = u_0$. Linearity of $\overline{A}$ implies $u_0 = 0$. This contradiction proves 2).

3) Since A is closable then

$$A^* \overset{1)}{=} \overline{A^*} \overset{2)}{=} (A^*)^{**} = (A)^{***} = (A^{**})^* \overset{2)}{=} \left(\overline{A}\right)^* .$$

This finishes the proof.

□

Example 2.4. Consider the Hilbert space $H = L^2(\mathbb{R})$ and the operator

$$Au(x) = (u, f_0)u_0(x),$$

where $u_0 \not\equiv 0, u_0 \in L^2(\mathbb{R})$ is fixed and $f_0 \neq 0$ is an arbitrary but fixed constant. We consider A on the domain

$$D(A) = \left\{u \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |f_0 u(x)| dx < \infty\right\} = L^2(\mathbb{R}) \cap L^1(\mathbb{R}).$$

It is known that $\overline{L^2(\mathbb{R}) \cap L^1(\mathbb{R})} = L^2(\mathbb{R})$. Thus A is densely defined. Let v be an element of $D(A^*)$. Then

$$(Au, v) = ((u, f_0)u_0, v) = (u, f_0)(u_0, v) = \left(u, \overline{(u_0, v)} f_0\right) = (u, (v, u_0) f_0) .$$

It means that

$$A^* v = (v, u_0) f_0.$$

But $(v, u_0) f_0$ must belong to $L^2(\mathbb{R})$. Since $(v, u_0) f_0$ is a constant and $f_0 \neq 0$ then (v, u_0) must be equal to 0. Thus

$$u_0 \perp D(A^*)$$

which implies that

$$u_0 \perp \overline{D(A^*)}.$$

Since $u_0 \neq 0$ then $\overline{D(A^*)} \neq H$. Thus A^* exists but is not densely defined.

Exercise 11. Assume that A is closable. Prove that $D(\overline{A})$ can be obtained as the closure of $D(A)$ by the norm

$$\left(\|Au\|^2 + \|u\|^2\right)^{1/2} .$$

Definition. Let $A : H \to H$ with $\overline{D(A)} = H$. We say that A is

1. *symmetric* if $A \subset A^*$;
2. *self-adjoint* if $A = A^*$;
3. *essentially self-adjoint* if $\left(\overline{A}\right)^* = \overline{A}$.

Remark. A symmetric operator is always closable and its closure is also symmetric. Indeed, if $A \subset A^*$ then $D(A) \subset D(A^*)$. Hence

$$H = \overline{D(A)} \subset \overline{D(A^*)} \subset H$$

implies that $\overline{D(A^*)} = H$. That's why A is closable. Since $\overline{A}$ is the smallest closed extension of A then

$$A \subset \overline{A} \subset A^* = \left(\overline{A}\right)^*$$

i.e. $\overline{A}$ is also symmetric.

Some properties of symmetric operator A are:

1) $A \subset \overline{A} = A^{**} \subset A^*$
2) $A = \overline{A} = A^{**} \subset A^*$ if A is closed.
3) $A = \overline{A} = A^{**} = A^*$ if A is self-adjoint.
4) $A \subset \overline{A} = A^{**} = A^*$ if A is essentially self-adjoint.

Theorem 2 (J. von Neumann)**.** *Assume that $A \subset A^*$.*

1) If $D(A) = H$ then $A = A^*$ and bounded.

2) If $R(A) = H$ then $A = A^*$ and A^{-1} exists and is bounded.

3) If A^{-1} exists then $A = A^*$ if and only if $A^{-1} = (A^{-1})^*$.

Proof. 1) Since $A \subset A^*$ then $H = D(A) \subset D(A^*) \subset H$ and hence $D(A) = D(A^*) = H$. Thus $A = A^*$ and the Hellinger-Toeplitz theorem (Exercise 9) says that A is bounded.

2,3) Let us assume that $u_0 \in D(A)$ and $Au_0 = 0$. Then for any $v \in D(A)$ we obtain that

$$0 = (Au_0, v) = (u_0, Av).$$

It means that $u_0 \perp H$ and therefore $u_0 = 0$. It follows that A^{-1} exists and $D(A^{-1}) = R(A) = H$. Hence $(A^{-1})^*$ exists. Let us prove that $(A^*)^{-1}$ exists too and $(A^*)^{-1} = (A^{-1})^*$. Indeed, if $u \in D(A)$ and $v \in D\left((A^{-1})^*\right)$ then

$$(u, v) = (A^{-1}Au, v) = (Au, \left(A^{-1}\right)^* v).$$

This equality implies that

$$\left(A^{-1}\right)^* v \in D(A^*)$$

and

$$A^* \left(A^{-1}\right)^* v = v. \tag{2.1}$$

Similarly, if $u \in D(A^{-1})$ and $v \in D(A^*)$ then

$$(u, v) = (AA^{-1}u, v) = (A^{-1}u, A^*v)$$

and therefore

$$A^*v \in D\left(\left(A^{-1}\right)^*\right)$$

and

$$\left(A^{-1}\right)^* A^*v = v. \tag{2.2}$$

It follows from (2.1) and (2.2) that $(A^*)^{-1}$ exists and $(A^*)^{-1} = (A^{-1})^*$.

Exercise 12. Let A and B be injective operators. Prove that if $A \subset B$ then $A^{-1} \subset B^{-1}$.

Since $A \subset A^*$ we have by Exercise 12 that

$$A^{-1} \subset (A^*)^{-1} = \left(A^{-1}\right)^*$$

i.e. A^{-1} is also symmetric. But $D(A^{-1}) = H$. That's why we may conclude that $H = D(A^{-1}) \subset D\left((A^{-1})^*\right) \subset H$ and hence $D(A^{-1}) = D\left((A^{-1})^*\right) = H$. Thus A^{-1} is self-adjoint and bounded (Hellinger-Toeplitz theorem). Finally,

$$A^{-1} = \left(A^{-1}\right)^* = (A^*)^{-1}$$

if and only if $A = A^*$.

□

Theorem 3 (Basic criterion of self-adjointness)**.** *If $A \subset A^*$ then the following statements are equivalent:*

1) $A = A^*$.

2) $A = \overline{A}$ and $N(A^* \pm iI) = \{0\}$.

3) $R(A \pm iI) = H$.

Proof. **1)⇒2)** Since $A = A^*$ then A is closed. Suppose that $u_0 \in N(A^* - iI)$ i.e. $u_0 \in D(A^*) = D(A)$ and $Au_0 = iu_0$. Then

$$i(u_0, u_0) = (iu_0, u_0) = (Au_0, u_0) = (u_0, Au_0) = (u_0, iu_0) = -i(u_0, u_0).$$

This implies that $u_0 = 0$ i.e. $N(A^* - iI) = \{0\}$. The proof of $N(A^* + iI) = \{0\}$ is left to the reader.

2)⇒3) Since $A = \overline{A}$ and $N(A^* \pm iI) = \{0\}$ then, for example, the equation $A^*u = -iu$ has only the trivial solution $u = 0$. It implies that $\overline{R(A - iI)} = H$. For otherwise there exists $u_0 \neq 0$ such that $u_0 \perp R(A - iI)$. It means that for any $u \in D(A)$ we have

$$((A - iI)u, u_0) = 0$$

and therefore $u_0 \in D(A^* + iI)$ and $(A^* + iI)u_0 = 0$ or $A^*u_0 = -iu_0, u_0 \neq 0$. This contradiction proves that $\overline{R(A - iI)} = H$. Next, since A is closed then $\Gamma(A)$ is also closed and due to the fact that A is symmetric we have

$$\begin{aligned} \|(A - iI)u\|^2 &= ((A - iI)u, (A - iI)u) = \|Au\|^2 - i(u, Au) + i(Au, u) + \|u\|^2 \\ &= \|Au\|^2 + \|u\|^2, \quad u \in D(A). \end{aligned}$$

That's why if $(A - iI)u_n \to v_0$ then Au_n and u_n are convergent i.e. $Au_n \to v_0', u_n \to u_0'$ and $u_n \in D(A)$. The closedness of A implies that $u_0' \in D(A)$ and $v_0' = Au_0'$ i.e. $(A - iI)u_n \to Au_0' - iu_0' = v_0$. It means that $R(A - iI)$ is a closed set i.e. $\overline{R(A - iI)} = R(A - iI) = H$. The proof of $R(A + iI) = H$ is left to the reader.

3)⇒1) Assume that $R(A \pm iI) = H$. Since $A \subset A^*$ it suffices to show that $D(A^*) \subset D(A)$. For every $u \in D(A^*)$ we have $(A^* - iI)u \in H$. Part 3) implies that there exists $v_0 \in D(A)$ such that

$$(A - iI)v_0 = (A^* - iI)u.$$

It is clear that $u - v_0 \in D(A^*)$ (since $A \subset A^*$) and

$$\begin{aligned} (A^* - iI)(u - v_0) &= (A^* - iI)u - (A^* - iI)v_0 = (A^* - iI)u - (A - iI)v_0 \\ &= (A - iI)v_0 - (A - iI)v_0 = 0. \end{aligned}$$

Hence $u - v_0 \in N(A^* - iI)$.

Exercise 13. Let A be a linear and densely defined operator in the Hilbert space H. Prove that

$$H = N(A^*) \oplus \overline{R(A)}.$$

By this exercise we know that

$$H = N(A^* - iI) \oplus \overline{R(A + iI)}.$$

But in our case $R(A + iI) = H$. Hence $N(A^* - iI) = \{0\}$ and therefore $u = v_0$. Thus $D(A) = D(A^*)$.

□

Exercise 14. Let $H = L^2(0, 1), A := i\frac{d}{dt}$ and

$$D_\gamma(A) = \left\{f \in L^2(0, 1) : f' \in L^2(0, 1), f(0) = f(1)e^{i\gamma}, \gamma \in \mathbb{R}\right\}.$$

Prove that A is self-adjoint on $D_\gamma(A)$.

3 J. von Neumann's spectral theorem

Definition. A bounded linear operator P on a Hilbert space H which is self-adjoint and *idempotent* i.e. $P^2 = P$ is called an orthogonal projection operator or a *projector*.

Proposition 1. *Let P be a projector.*

1) $\|P\| = 1$ *if* $P \neq 0$.

2) P *is a projector if and only if* $P^{\perp} := I - P$ *is a projector.*

3) $H = R(P) \oplus R(P^{\perp})$, $P|_{R(P)} = I$ *and* $P|_{R(P^{\perp})} = 0$.

4) *There is one-to-one correspondence between projectors on H and closed linear subspaces of H. More precisely, if $M \subset H$ is a closed linear subspace then there exists a projector $P_M : H \to M$ and, conversely, if $P : H \to H$ is a projector then $R(P)$ is a closed linear subspace.*

5) *If* $\{e_j\}_{j=1}^N, N \leq \infty$ *is an orthonormal system then*

$$P_N x := \sum_{j=1}^{N} (x, e_j) e_j, \quad x \in H$$

is a projector.

Proof. 1) Since $P = P^*$ and $P = P^2$ then $P = P^*P$. Hence $\|P\| = \|P^*P\|$. But $\|P^*P\| = \|P\|^2$. Indeed,

$$\|P^*P\| \leq \|P^*\| \, \|P\| \leq \|P\|^2$$

and

$$\begin{aligned} \|P\|^2 &= \sup_{\|x\|=1} \|Px\|^2 = \sup_{\|x\|=1} (Px, Px) = \sup_{\|x\|=1} (P^*Px, x) \leq \sup_{\|x\|=1} \|P^*Px\| \\ &= \|P^*P\| . \end{aligned}$$

Therefore $\|P\| = \|P\|^2$ or $\|P\| = 1$ if $P \neq 0$.

2) Since P is linear and bounded then the same is true about $I - P$. Moreover,

$$(I - P)^* = I - P^* = I - P$$

and

$$(I - P)^2 = (I - P)(I - P) = I - 2P + P^2 = I - P.$$

3) It follows immediately from $I = P + P^{\perp}$ that every $x \in H$ is of the form $u + v$, where $u \in R(P)$ and $v \in R(P^{\perp})$. Let us prove that $R(P) = \left(R(P^{\perp})\right)^{\perp}$. First assume that $w \in \left(R(P^{\perp})\right)^{\perp}$ i.e. $(w, (I - P)x) = 0$ for all $x \in H$. This is equivalent to
$$(w, x) = (w, Px) = (Pw, x), \quad x \in H$$
or $Pw = w$. Hence $w \in R(P)$ and so we have proved that $\left(R(P^{\perp})\right)^{\perp} \subset R(P)$. For the opposite embedding we let $w \in R(P)$. Then there exists $x_w \in H$ such that $w = Px_w$. If $z \in R(P^{\perp})$ then $z = P^{\perp}x_z = (I - P)x_z$ for some $x_z \in H$. Thus
$$(w, z) = (Px_w, (I - P)x_z) = (Px_w, x_z) - (Px_w, Px_z) = 0$$
since P is a projector. Therefore $w \in \left(R(P^{\perp})\right)^{\perp}$ and we may conclude that $R(P) = \left(R(P^{\perp})\right)^{\perp}$. This fact allows us to conclude that $R(P) = \overline{R(P)}$ and $H = R(P) \oplus R(P^{\perp})$. Moreover, it is easy to check by definition that $P|_{R(P)} = I$ and $P|_{R(P^{\perp})} = 0$.

4) If $M \subset H$ is a closed subspace then Theorem 1 in Section 1 implies that $x = u + v \in H$, where $u \in M$ and $v \in M^{\perp}$. In that case let us define $P_M : H \to M$ as
$$P_M x = u.$$
It is clear that $P_M^2 x = P_M u = u = P_M x$ i.e. $P_M^2 = P_M$. Moreover, if $y \in H$ then $y = u_1 + v_1, u_1 \in M, v_1 \in M^{\perp}$ and
$$(P_M x, y) = (u, u_1 + v_1) = (u, u_1) = (u + v, u_1) = (u + v, P_M y) = (x, P_M y)$$
i.e. $P_M^* = P_M$. Hence P_M is a projector. If P is a projector then we know from part 3) that $M := R(P)$ is closed subspace of H.

5) Let us assume that $N = \infty$. Define M as
$$M := \left\{ x \in H : x = \sum_{j=1}^{\infty} c_j e_j, \sum_{j=1}^{\infty} |c_j|^2 < \infty \right\}.$$
Then M is a closed subspace of H. If we define a linear operator P_M as
$$P_M x := \sum_{j=1}^{\infty} (x, e_j) e_j, \quad x \in H$$
then by Bessel's inequality we obtain that $P_M x \in M$ and
$$\|P_M x\| \le \|x\|.$$

It means that P_M is a bounded linear operator into M. But $P_M e_j = e_j$ and thus $P_M^2 x = P_M x$ for all $x \in H$. Next, for all $x, y \in H$ we have

$$\begin{aligned}(P_M x, y) &= \left(\sum_{j=1}^{\infty}(x, e_j)e_j, y\right) = \sum_{j=1}^{\infty}(x, e_j)\,(e_j, y) = \sum_{j=1}^{\infty}(x, (y, e_j)e_j) \\ &= \left(x, \sum_{j=1}^{\infty}(y, e_j)e_j\right) = (x, P_M y)\end{aligned}$$

i.e. $P_M^* = P_M$. The case of finite N requires no convergence questions and is left to the reader.

□

Definition. A bounded linear operator A on the Hilbert space H is smaller than or equal to a bounded operator B on H if

$$(Ax, x) \leq (Bx, x), \quad x \in H.$$

We denote this fact by $A \leq B$. We say that A is *non-negative* if $A \geq 0$. We denote $A > 0$, and say that A is *positive*, if $A \geq c_0 I$ for some $c_0 > 0$.

Remark. In the frame of this definition (Ax, x) and (Bx, x) must be real for all $x \in H$.

Proposition 2. *For two projectors P and Q the following statements are equivalent:*

1) *$P \leq Q$.*
2) *$\|Px\| \leq \|Qx\|$ for all $x \in H$.*
3) *$R(P) \subset R(Q)$.*
4) *$P = PQ = QP$.*

Proof. **1)⇔2)** Follows immediately from $(Px, x) = (P^2 x, x) = (Px, Px) = \|Px\|^2$.

3)⇔4) Assume $R(P) \subset R(Q)$. Then $QPx = Px$ or $QP = P$. Conversely, if $QP = P$ then clearly $R(P) \subset R(Q)$. Finally, $P = QP = P^* = (QP)^* = P^* Q^* = PQ$.

2)⇔4) If 4) holds then $Px = PQx$ and $\|Px\| = \|PQx\| \leq \|Qx\|$ for all $x \in H$. Conversely, if $\|Px\| \leq \|Qx\|$ then $Px = QPx + Q^{\perp}Px$ implies that

$$\|Px\|^2 = \|QPx\|^2 + \left\|Q^{\perp}Px\right\|^2 \leq \|QPx\|^2.$$

Hence

$$\left\|Q^{\perp}Px\right\|^2 = 0$$

i.e. $Q^{\perp}Px = 0$ for all $x \in H$. Hence $P = QP = PQ$.

□

Exercise 15. Let $\{P_j\}_{j=1}^{\infty}$ be a sequence of projectors with $P_j \leq P_{j+1}$ for each $j = 1, 2, \dots$. Prove that $\lim_{j\to\infty} P_j := P$ exists and that P is a projector.

Definition. Any linear map $A : H \to H$ with the property

$$\|Ax\| = \|x\|, \quad x \in H$$

is called an *isometry*.

Exercise 16. Prove that

1) A is an isometry if and only if $A^*A = I$.

2) Every isometry A has an inverse $A^{-1} : R(A) \to H$ and $A^{-1} = A^*|_{R(A)}$.

3) If A is an isometry then AA^* is a projector on $R(A)$.

Definition. A surjective isometry $U : H \to H$ is called a *unitary operator*.

Remark. It follows that U is unitary if and only if it is surjective and $U^*U = UU^* = I$ i.e. $(Ux, Uy) = (x, y)$ for all $x, y \in H$.

Definition. Let H be a Hilbert space. The family of operators $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$ is called a *spectral family* if the following conditions are satisfied:

1) E_λ is a projector for all $\lambda \in \mathbb{R}$.

2) $E_\lambda \leq E_\mu$ for all $\lambda < \mu$.

3) $\{E_\lambda\}$ is right continuous with respect to the strong operator topology i.e.

$$\lim_{s\to t+0} \|E_s x - E_t x\| = 0$$

for all $x \in H$.

4) $\{E_\lambda\}$ is normalized as follows:

$$\lim_{\lambda\to-\infty} \|E_\lambda x\| = 0, \quad \lim_{\lambda\to+\infty} \|E_\lambda x\| = \|x\|$$

for all $x \in H$. The latter condition can also be formulated as

$$\lim_{\lambda\to+\infty} \|E_\lambda x - x\| = 0.$$

Remark. It follows from the previous definition and Proposition 2 that

$$E_\lambda E_\mu = E_{\min\{\lambda,\mu\}}.$$

Proposition 3. *For every fixed $x, y \in H$, $(E_\lambda x, y)$ is a function of bounded variation with respect to $\lambda \in \mathbb{R}$.*

Proof. Let us define

$$E(\alpha,\beta] := E_\beta - E_\alpha, \quad \alpha < \beta.$$

Then $E(\alpha,\beta]$ is a projector. Indeed,

$$E(\alpha,\beta]^* = E_\beta^* - E_\alpha^* = E_\beta - E_\alpha = E(\alpha,\beta]$$

i.e. $E(\alpha,\beta]$ is self-adjoint. It is also idempotent due to

$$\begin{aligned}(E(\alpha,\beta])^2 &= (E_\beta - E_\alpha)(E_\beta - E_\alpha) = E_\beta^2 - E_\alpha E_\beta - E_\beta E_\alpha + E_\alpha^2 \\ &= E_\beta - E_\alpha - E_\alpha + E_\alpha = E(\alpha,\beta].\end{aligned}$$

Another property is that

$$E(\alpha_1,\beta_1]x \perp E(\alpha,\beta]y, \quad x,y \in H$$

if $\beta_1 \leq \alpha$ or $\beta \leq \alpha_1$. To see this for $\beta_1 \leq \alpha$ calculate

$$\begin{aligned}(E(\alpha_1,\beta_1]x, E(\alpha,\beta]y) &= (E_{\beta_1}x - E_{\alpha_1}x, E_\beta y - E_\alpha y) \\ &= (E_{\beta_1}x, E_\beta y) - (E_{\alpha_1}x, E_\beta y) - (E_{\beta_1}x, E_\alpha y) + (E_{\alpha_1}x, E_\alpha y) \\ &= (x, E_{\beta_1}y) - (x, E_{\alpha_1}y) - (x, E_{\beta_1}y) + (x, E_{\alpha_1}y) = 0.\end{aligned}$$

Let now

$$\lambda_0 < \lambda_1 < \cdots < \lambda_n.$$

Then

$$\begin{aligned}\sum_{j=1}^n \left|\left(E_{\lambda_j}x, y\right) - \left(E_{\lambda_{j-1}}x, y\right)\right| &= \sum_{j=1}^n |(E(\lambda_{j-1},\lambda_j]x, y)| \\ &= \sum_{j=1}^n |(E(\lambda_{j-1},\lambda_j]x, E(\lambda_{j-1},\lambda_j]y)| \\ &\leq \sum_{j=1}^n \|E(\lambda_{j-1},\lambda_j]x\| \, \|E(\lambda_{j-1},\lambda_j]y\| \\ &\leq \left(\sum_{j=1}^n \|E(\lambda_{j-1},\lambda_j]x\|^2\right)^{1/2} \left(\sum_{j=1}^n \|E(\lambda_{j-1},\lambda_j]y\|^2\right)^{1/2} \\ &= \left\|\sum_{j=1}^n E(\lambda_{j-1},\lambda_j]x\right\| \left\|\sum_{j=1}^n E(\lambda_{j-1},\lambda_j]y\right\| \\ &= \|E(\lambda_0,\lambda_n]x\| \, \|E(\lambda_0,\lambda_n]y\| \leq \|x\| \, \|y\| .\end{aligned}$$

Here we have made use of orthogonality, normalization and the Cauchy-Schwarz-Bunjakovskii inequality. □

Due to Proposition 3 we can define a Stieltjes integral. Moreover, for any continuous function $f(\lambda)$ we may conclude that the limit

$$\lim_{\Delta\to 0}\sum_{j=1}^{n} f(\lambda_j^*)\left(E(\lambda_{j-1},\lambda_j]x,y\right) = \lim_{\Delta\to 0}\left(\sum_{j=1}^{n} f(\lambda_j^*)E(\lambda_{j-1},\lambda_j]x,y\right),$$

where $\lambda_j^* \in [\lambda_{j-1},\lambda_j]$, $\alpha=\lambda_0<\lambda_1<\cdots<\lambda_n=\beta$ and $\Delta=\max_{1\le j\le n}|\lambda_{j-1}-\lambda_j|$ exists and by definition this limit is

$$\int_\alpha^\beta f(\lambda)d(E_\lambda x,y), \quad x,y\in H.$$

It can be shown that this is equivalent to the existence of the limit in H

$$\lim_{\Delta\to 0}\sum_{j=1}^{n} f(\lambda_j^*)E(\lambda_{j-1},\lambda_j]x,$$

which we denote by

$$\int_\alpha^\beta f(\lambda)dE_\lambda x.$$

Thus

$$\int_\alpha^\beta f(\lambda)d(E_\lambda x,y) = \left(\int_\alpha^\beta f(\lambda)dE_\lambda x,y\right), \quad x,y\in H.$$

For the spectral representation of self-adjoint operators one needs not only integrals over finite intervals but also over whole line which is naturally defined as the limit

$$\int_{-\infty}^{\infty} f(\lambda)d(E_\lambda x,y) = \lim_{\substack{\alpha\to-\infty\\ \beta\to\infty}}\int_\alpha^\beta f(\lambda)d(E_\lambda x,y) = \left(\int_{-\infty}^{\infty} f(\lambda)dE_\lambda x,y\right)$$

if it exists. Deriving first some basic properties of the integral just defined one can check that

$$\int_{-\infty}^{\infty} f(\lambda)d(E_\lambda E_\beta x,y) = \int_{-\infty}^{\beta} f(\lambda)d(E_\lambda x,y) := \lim_{\alpha\to-\infty}\int_\alpha^\beta f(\lambda)d(E_\lambda x,y), \quad x,y\in H.$$

Theorem 1. *Let $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$ be a spectral family on the Hilbert space H and let f be a real-valued continuous function on the line. Define*

$$D := \left\{x\in H : \int_{-\infty}^{\infty}|f(\lambda)|^2 d(E_\lambda x,x)<\infty\right\}$$

(or $D := \left\{x\in H : \int_{-\infty}^{\infty} f(\lambda)dE_\lambda x \text{ exists}\right\}$). Let us define on this domain an operator A as

$$(Ax,y) = \int_{-\infty}^{\infty} f(\lambda)d(E_\lambda x,y), \quad x\in D(A):=D, y\in H$$

(or $Ax=\int_{-\infty}^{\infty} f(\lambda)dE_\lambda x, x\in D(A)$). Then A is self-adjoint and satisfies

$$E(\alpha,\beta]A\subset AE(\alpha,\beta], \quad \alpha<\beta.$$

Proof. It can be shown that the integral

$$\int_{-\infty}^{\infty} f(\lambda) d\left(E_\lambda x, y\right)$$

exists for $x \in D$ and $y \in H$. Thus (Ax, y) is well-defined. Let v be any element of H and let $\varepsilon > 0$. Then, by normalization, there exists $\alpha < -R$ and $\beta > R$ with R large enough such that

$$\|v - E(\alpha, \beta]v\| = \|v - E_\beta v + E_\alpha v\| \leq \|(I - E_\beta)v\| + \|E_\alpha v\| < \varepsilon.$$

On the other hand,

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(\lambda)|^2 d(E_\lambda E(\alpha,\beta]v, E(\alpha,\beta]v) &= \int_{-\infty}^{\infty} |f(\lambda)|^2 d(E_\lambda E(\alpha,\beta]v, v) \\
&= \int_{-\infty}^{\infty} |f(\lambda)|^2 d(E_\lambda E_\beta v, v) - \int_{-\infty}^{\infty} |f(\lambda)|^2 d(E_\lambda E_\alpha v, v) \\
&= \int_{-\infty}^{\beta} |f(\lambda)|^2 d(E_\lambda v, v) - \int_{-\infty}^{\alpha} |f(\lambda)|^2 d(E_\lambda v, v) \\
&= \int_{\alpha}^{\beta} |f(\lambda)|^2 d(E_\lambda v, v) < \infty.
\end{aligned}$$

These two facts mean that $E(\alpha, \beta]v \in D$ and $\overline{D} = H$. Since $f(\lambda) = \overline{f(\lambda)}$ then A is symmetric. Indeed,

$$\begin{aligned}
(Ax, y) &= \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, y) = \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \int_{\alpha}^{\beta} f(\lambda) d(E_\lambda x, y) \\
&= \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \int_{\alpha}^{\beta} f(\lambda) d(x, E_\lambda y) = \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \left(x, \int_{\alpha}^{\beta} f(\lambda) dE_\lambda y\right) \\
&= \left(x, \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \int_{\alpha}^{\beta} f(\lambda) dE_\lambda y\right) = (x, Ay).
\end{aligned}$$

In order to prove that $A = A^*$ it remains to show that $D(A^*) \subset D(A)$. Let $u \in D(A^*)$. Then

$$(E(\alpha, \beta]z, A^* u) = (AE(\alpha, \beta]z, u) = \int_{\alpha}^{\beta} f(\lambda) d(E_\lambda z, u)$$

for any $z \in H$. This equality implies that

$$\begin{aligned}
(z, A^* u) &= \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \int_{\alpha}^{\beta} f(\lambda) d(E_\lambda z, u) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda z, u) = \int_{-\infty}^{\infty} f(\lambda) d(z, E_\lambda u) \\
&= \overline{\int_{-\infty}^{\infty} f(\lambda) d(E_\lambda u, z)} = \overline{(Au, z)} = (z, Au),
\end{aligned}$$

where the integral exists because (z, A^*u) exists. Hence $u \in D(A)$ and $A^*u = Au$. For the second claim we first calculate

$$\begin{aligned} E(\alpha,\beta]Ax &= (E_\beta - E_\alpha)\,Ax = (E_\beta - E_\alpha)\int_{-\infty}^{\infty} f(\lambda)dE_\lambda x \\ &= \int_{-\infty}^{\infty} f(\lambda)dE_\lambda E_\beta x - \int_{-\infty}^{\infty} f(\lambda)dE_\lambda E_\alpha x = \int_{-\infty}^{\beta} f(\lambda)dE_\lambda x - \int_{-\infty}^{\alpha} f(\lambda)dE_\lambda x \\ &= \int_{\alpha}^{\beta} f(\lambda)dE_\lambda x = \int_{-\infty}^{\infty} f(\lambda)dE_\lambda\,(E_\beta - E_\alpha)\,x = A\,(E_\beta - E_\alpha)\,x = AE(\alpha,\beta]x \end{aligned}$$

for any $x \in D(A)$. Since the left hand side is defined on $D(A)$ and the right hand side on all of H then the latter is an extension of the former. □

Exercise 17. Let A be as in Theorem 1. Prove that

$$\|Au\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 d(E_\lambda u, u)$$

if $u \in D(A)$.

Exercise 18. Let $H = L^2(\mathbb{R})$ and $Au(t) = tu(t), t \in \mathbb{R}$. Define $D(A)$ on which $A = A^*$ and evaluate the spectral family $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$.

Theorem 2 (J. von Neumann's spectral theorem)**.** *Every self-adjoint operator A on the Hilbert space H has a unique spectral representation i.e. there is a unique spectral family $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$ such that*

$$Ax = \int_{-\infty}^{\infty} \lambda dE_\lambda x, \quad x \in D(A)$$

(i.e. $(Ax, y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x, y), x \in D(A), y \in H$), where $D(A)$ is defined as

$$D(A) = \left\{x \in H : \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x, x) < \infty\right\}.$$

Proof. At first we assume that this theorem holds when A is bounded, that is, there is a unique spectral family $\{F_\mu\}_{\mu=-\infty}^{\infty}$ such that

$$Au = \int_{-\infty}^{\infty} \mu dF_\mu u, \quad u \in H$$

since $D(A) = H$ in this case. But $F_\mu \equiv 0$ for $\mu < m$ and $F_\mu \equiv I$ for $\mu > M$, where

$$m = \inf_{\|x\|=1}(Ax, x), \quad M = \sup_{\|x\|=1}(Ax, x).$$

That's why the spectral representation has a view

$$Au = \int_{m}^{M} \mu dF_\mu u, \quad u \in H.$$

Let us consider now an unbounded operator which is semibounded from below i.e.

$$(Au, u) \geq m_0(u, u), \quad u \in D(A)$$

with some constant m_0. We assume without loss of generality that $(Au, u) \geq (u, u)$. This condition implies that A^{-1} exists, is defined over whole H and $\|A^{-1}\| \leq 1$. Indeed, A^{-1} exists and is bounded because $Au = 0$ if and only if $u = 0$. The norm estimate follows from

$$(v, A^{-1}v) \geq \left\|A^{-1}v\right\|^2, \quad v \in D(A^{-1}).$$

Since A^{-1} is bounded then $D(A^{-1})$ is a closed subspace in H. But self-adjointness of A means that $A^{-1} = (A^{-1})^*$. That's why A^{-1} is closed and $\overline{D(A^{-1})} = H$ i.e. A^{-1} is densely defined. Therefore $D(A^{-1}) = H$ and $R(A) = H$. Since

$$0 \leq (A^{-1}v, v) \leq \|v\|^2, \quad v \in H$$

we may conclude in this case that $m \geq 0, M \leq 1$ and

$$A^{-1}v = \int_0^1 \mu dF_\mu v, \quad v \in H,$$

where $\{F_\mu\}$ is the spectral family of A^{-1}. Let us note that $F_1 = I$ and $F_0 = 0$. They follow from the spectral theorem and from the fact that $A^{-1}v = 0$ if and only if $v = 0$. Next, let us define the operator $B_\varepsilon, \varepsilon > 0$ as

$$B_\varepsilon u := \int_\varepsilon^1 \frac{1}{\mu} dF_\mu u, \quad u \in D(A).$$

For every $v \in H$ we have

$$\begin{aligned} B_\varepsilon A^{-1}v &= \int_\varepsilon^1 \frac{1}{\mu} dF_\mu (A^{-1}v) = \int_\varepsilon^1 \frac{1}{\mu} dF_\mu \left(\int_0^1 \lambda dF_\lambda v\right) = \int_\varepsilon^1 \frac{1}{\mu} d\left(\int_0^1 \lambda d(F_\mu F_\lambda v)\right) \\ &= \int_\varepsilon^1 \frac{1}{\mu} d\left(\int_\varepsilon^\mu \lambda dF_\lambda v\right) = \int_\varepsilon^1 \frac{1}{\mu} \mu dF_\mu v = \int_\varepsilon^1 dF_\mu v = F_1 v - F_\varepsilon v = v - F_\varepsilon v. \end{aligned}$$

Since every spectral family is right continuous then

$$\lim_{\varepsilon \to 0+0} B_\varepsilon A^{-1} v = v$$

exists. For every $u \in D(A)$ we have similarly,

$$A^{-1} B_\varepsilon u = \int_0^1 \mu dF_\mu (B_\varepsilon u) = \int_\varepsilon^1 \mu d\left(\int_\varepsilon^\mu \frac{1}{\lambda} dF_\lambda u\right) = u - F_\varepsilon u$$

and hence

$$\lim_{\varepsilon \to 0+0} A^{-1} B_\varepsilon u = u$$

exists. These two equalities mean that

$$\lim_{\varepsilon\to 0+0} B_\varepsilon = \left(A^{-1}\right)^{-1} = A$$

exists and the spectral representation

$$A = \int_0^1 \frac{1}{\mu} dF_\mu = \lim_{\varepsilon\to 0+} \int_\varepsilon^1 \frac{1}{\mu} dF_\mu$$

holds. If we define $E_\lambda = I - F_{\frac{1}{\lambda}}, 1 \le \lambda < \infty$ then

$$A = -\int_0^1 \frac{1}{\mu} dE_{\frac{1}{\mu}} = \int_1^\infty \lambda dE_\lambda.$$

Exercise 19. Prove that this $\{E_\lambda\}$ is a spectral family.

Domain $D(A)$ can be characterized as

$$D(A) = \left\{u \in H : \int_1^\infty \lambda^2 d(E_\lambda u, u) < \infty\right\} = \left\{u \in H : \int_0^1 \frac{1}{\mu^2} d(F_\mu u, u) < \infty\right\}.$$

This proves the theorem for self-adjoint operators that are semibounded from below. For bounded operators we will only sketch the proof.

Step 1. If $A = A^*$ and bounded then we can define

$$p_N(A) := a_0 I + a_1 A + \cdots + a_N A^N, \quad N \in \mathbb{N},$$

where $a_j \in \mathbb{R}$ for $j = 0, 1, \ldots, N$. Then $p_N(A)$ is also self-adjoint and bounded with

$$\|p_N(A)\| \le \sup_{|t|\le\|A\|} |p_N(t)|.$$

Step 2. For every continuous function f on $[m, M]$, where m and M are as above we can define $f(A)$ as an approximation by $p_N(A)$ i.e. we can prove that for any $\varepsilon > 0$ there exists $p_N(A)$ such that

$$\|f(A) - p_N(A)\| < \varepsilon.$$

Step 3. For every $u, v \in H$ let us define the functional L as

$$L(f) := (f(A)u, v).$$

Then

$$|L(f)| \le |(f(A)u, v)| \le \|f(A)\| \, \|u\| \, \|v\|$$

that is, $L(f)$ is a bounded linear functional on $C[m, M]$.

Step 4. (Riesz's theorem) Every positive linear continuous functional L on $C_0[a,b]$ can be represented in the form

$$L(f) = \int_a^b f(x) d\nu(x),$$

where ν is a measure that satisfies the conditions

1) $L(f) \geq 0$ for $f \geq 0$
2) $|L(f)| \leq \nu(K) \|f\|_K$, where $K \subset [a,b]$ is compact and

$$\|f\|_K = \max_{x \in K} |f(x)|.$$

Step 5. It follows from Step 4 that

$$(Au, v) = \int_m^M \lambda d\nu(\lambda; u, v).$$

Step 6. It is possible to prove that $\nu(\lambda; u, v)$ is a self-adjoint bilinear form. That's why we may conclude that there exists a self-adjoint and bounded operator E_λ such that

$$\nu(\lambda; u, v) = (E_\lambda u, v).$$

This operator is idempotent and we may define $E_\lambda \equiv 0$ for $\lambda < m$ and $E_\lambda \equiv I$ for $\lambda \geq M$. Thus $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$ is the required spectral family and this theorem is proved.

□

Let $A : H \to H$ be a self-adjoint operator in the Hilbert space H. Then by J. von Neumann's spectral theorem we can write

$$Au = \int_{-\infty}^{\infty} \lambda d E_\lambda u, \quad u \in D(A).$$

For every continuous function f we can define

$$D_f := \left\{ u \in H : \int_{-\infty}^{\infty} |f(\lambda)|^2 d(E_\lambda u, u) < \infty \right\}.$$

This set is a linear subspace of H. For every $u \in D_f$ and $v \in H$ let us define the linear functional

$$L(v) := \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda u, v) = \left(\int_{-\infty}^{\infty} f(\lambda) d E_\lambda u, v \right).$$

This functional is continuous because it is bounded. Indeed,

$$|L(v)|^2 \leq \left\| \int_{-\infty}^{\infty} f(\lambda) d E_\lambda u \right\|^2 \|v\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 d(E_\lambda u, u) \|v\|^2 = c(u) \|v\|^2.$$

By the Riesz-Frechet theorem this functional can be expressed in the form of an inner product i.e. there exists $z \in H$ such that

$$\int_{-\infty}^{\infty} f(\lambda) d(E_\lambda u, v) = (z, v), \quad v \in H.$$

We set

$$z := f(A)u, \quad u \in D_f$$

i.e.

$$(f(A)u, v) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda u, v).$$

Remark. Since in general f is not real-valued then $f(A)$ is not a self-adjoint operator in general.

Example 3.1. Consider

$$f(\lambda) = \frac{\lambda - i}{\lambda + i}, \quad \lambda \in \mathbb{R}$$

Denote

$$U_A := f(A) = \int_{-\infty}^{\infty} \frac{\lambda - i}{\lambda + i} dE_\lambda.$$

The operator U_A is called the *Cayley transform*. Since $|f(\lambda)| = 1$ then $D_f = D(U_A) = H$ and

$$\begin{aligned} \|U_A u\|^2 &= \int_{-\infty}^{\infty} |f(\lambda)|^2 d(E_\lambda u, u) = \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \int_\alpha^\beta d(E_\lambda u, u) = \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} ((E_\beta u, u) - (E_\alpha u, u)) \\ &= \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \left(\|E_\beta u\|^2 - \|E_\alpha u\|^2 \right) = \|u\|^2 \end{aligned}$$

by normalization of $\{E_\lambda\}$. Hence U_A is an isometry. There is one-to-one correspondence between self-adjoint operators and their Cayley transforms. Indeed,

$$U_A = (A - iI)(A + iI)^{-1}$$

is equivalent to

$$\begin{cases} I - U_A = 2i(A + iI)^{-1} \\ I + U_A = 2A(A + iI)^{-1} \end{cases}$$

or

$$A = i(I + U_A)(I - U_A)^{-1}.$$

Example 3.2. Consider

$$f(\lambda) = \frac{1}{\lambda - z}, \quad \lambda \in \mathbb{R}, z \in \mathbb{C}, \operatorname{Im} z \neq 0.$$

Denote

$$R_z := (A - zI)^{-1} = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} dE_\lambda.$$

The operator R_z is called the *resolvent* of A. Since

$$\left|\frac{1}{\lambda - z}\right| \le \frac{1}{|\operatorname{Im} z|}$$

for all $\lambda \in \mathbb{R}$ then R_z is bounded and defined on whole H.

Exercise 20. Let $A = A^*$ with spectral family E_λ. Let $u \in D(f(A))$ and $v \in D(g(A))$. Prove that

$$(f(A)u, g(A)v) = \int_{-\infty}^{\infty} f(\lambda)\overline{g(\lambda)} d(E_\lambda u, v).$$

Exercise 21. Let $A = A^*$ with spectral family E_λ. Let $u \in D(f(A))$. Prove that $f(A)u \in D(g(A))$ if and only if $u \in D((gf)(A))$ and that

$$(gf)(A)u = \int_{-\infty}^{\infty} g(\lambda)f(\lambda)dE_\lambda u.$$

Remark. It follows from Exercise 21 that

$$(gf)(A) = (fg)(A)$$

on the domain $D((fg)(A)) \cap D((gf)(A))$.

4 Spectrum of self-adjoint operators

Definition. Given a linear operator A in the Hilbert space H with domain $D(A)$, $\overline{D(A)} = H$, the set

$$\rho(A) = \left\{z \in \mathbb{C} : (A - zI)^{-1} \text{ exists as a bounded operator from } H \text{ to } D(A)\right\}$$

is called the *resolvent set* of A. Its complement

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

is called the *spectrum* of A.

Theorem 1. *1) If $A = \overline{A}$ then the resolvent set is open and the resolvent operator $R_z := (A - zI)^{-1}$ is an analytic function from $\rho(A)$ to $B(H; H)$, the set of all linear operators in H. Furthermore, the* resolvent identity

$$R_z - R_\xi = (z - \xi) R_z R_\xi, \quad z, \xi \in \rho(A)$$

holds and $R_z' = (R_z)^2$.

2) If $A = A^*$ then $z \in \rho(A)$ if and only if there exists $C_z > 0$ such that

$$\|(A - zI)u\| \geq C_z \|u\|$$

for all $u \in D(A)$.

Proof. 1) Assume that $z_0 \in \rho(A)$. Then R_{z_0} is a bounded linear operator from H to $D(A)$ and thus $r := \|R_{z_0}\|^{-1} > 0$. Let us define for $|z - z_0| < r$ the operator

$$G_{z_0} := (z - z_0) R_{z_0}.$$

Then G_{z_0} is bounded with $\|G_{z_0}\| < 1$. Hence it defines the operator

$$(I - G_{z_0})^{-1} = \sum_{j=0}^{\infty} (G_{z_0})^j$$

because this Neumann series converges. But for $|z - z_0| < r$ we have

$$A - zI = (A - z_0 I)(I - G_{z_0})$$

or

$$(A - zI)^{-1} = (I - G_{z_0})^{-1} R_{z_0}.$$

Hence R_z exists with $D(R_z) = H$ and is bounded. It remains to show that $R(R_z) \subset D(A)$. For $x \in H$ we know that

$$y := (A - zI)^{-1} x \in H.$$

We claim that $y \in D(A)$. Indeed,

$$\begin{aligned} y &= (A - zI)^{-1}x = (I - G_{z_0})^{-1} R_{z_0} x = \sum_{j=0}^{\infty} (z - z_0)^j \left(R_{z_0}\right)^{j+1} x \\ &= \lim_{n\to\infty} \sum_{j=0}^{n} (z - z_0)^j \left(R_{z_0}\right)^{j+1} x. \end{aligned}$$

It follows from this representation that $R_z = (A - zI)^{-1}$ is an analytic function from $\rho(A)$ to $B(H; H)$. Next we denote

$$s_n x := \sum_{j=0}^{n} (z - z_0)^j \left(R_{z_0}\right)^{j+1} x.$$

It is clear that $s_n x \in D(A)$ and that $\lim_{n\to\infty} s_n x = y$. Moreover,

$$\lim_{n\to\infty} (A - zI) s_n x = x.$$

Denoting $y_n := s_n x$ we may conclude from the criterion for closedness that

$$\begin{cases} y_n \in D(A) \\ y_n \to y \\ (A - zI) y_n \to x \end{cases} \quad \Rightarrow \quad \begin{cases} y \in D(A) \\ x = (A - zI) y. \end{cases}$$

Hence $y = (A - zI)^{-1} x \in D(A)$ and therefore $\rho(A)$ is open. The resolvent identity is proved by straightforward calculation

$$\begin{aligned} R_z - R_\xi &= R_z (A - \xi I) R_\xi - R_z (A - zI) R_\xi = R_z \left[(A - \xi I) - (A - zI)\right] R_\xi \\ &= (z - \xi) R_z R_\xi. \end{aligned}$$

Finally, the limit

$$\lim_{z\to\xi} \frac{R_z - R_\xi}{z - \xi} = \lim_{z\to\xi} R_z R_\xi = (R_z)^2$$

exists and hence $R_z' = (R_z)^2$ exists. It proves this part.

2) Assume that $A = A^*$. If $z \in \rho(A)$ then by definition R_z maps from H to $D(A)$. Hence there exists $M_z > 0$ such that

$$\|R_z v\| \le M_z \|v\|, \quad v \in H.$$

Since $u = R_z (A - zI) u$ for any $u \in D(A)$ then we get

$$\|u\| \le M_z \|(A - zI) u\|, \quad u \in D(A).$$

This is equivalent to

$$\|(A - zI) u\| \ge \frac{1}{M_z} \|u\|, \quad u \in D(A).$$

Conversely, if there exists $C_z > 0$ such that

$$\|(A - zI)u\| \geq C_z \|u\|, \quad u \in D(A).$$

then $(A - zI)^{-1}$ is bounded. Since A is self-adjoint then $(A - zI)^{-1}$ is defined over whole H. Indeed, if $R(A - zI) \neq H$ then there exists $v_0 \neq 0$ such that $v_0 \perp R(A - zI)$. This means that

$$(v_0, (A - zI)u) = 0, \quad u \in D(A)$$

or

$$(Au, v_0) = (zu, v_0)$$

or

$$(u, A^* v_0) = (u, \overline{z} v_0).$$

Thus $v_0 \in D(A^*)$ and $A^* v_0 = \overline{z} v_0$. Since $A = A^*$ then $v_0 \in D(A)$ and $A v_0 = \overline{z} v_0$ or

$$(A - \overline{z} I) v_0 = 0.$$

It is easy to check that $\|(A - \overline{z}I)u\|^2 = \|(A - zI)u\|^2$ for any $u \in D(A)$. Therefore

$$\|(A - \overline{z}I)v_0\| = \|(A - zI)v_0\| \geq C_z \|v_0\|.$$

Hence $v_0 = 0$ and $D\left((A - zI)^{-1}\right) = R(A - zI) = H$. It means that $z \in \rho(A)$.

□

Corollary 1. *If $A = A^*$ then $\sigma(A) \neq \emptyset, \sigma(A) = \overline{\sigma(A)}$ and $\sigma(A) \subset \mathbb{R}$.*

Proof. If $z = \alpha + i\beta \in \mathbb{C}$ with $\operatorname{Im} z = \beta \neq 0$ then

$$\|(A - zI)x\|^2 = \|(A - \alpha I)x - i\beta x\|^2 = \|(A - \alpha I)x\|^2 + |\beta|^2 \|x\|^2 \geq |\beta|^2 \|x\|^2.$$

It implies (see part 2) of Theorem 1) that $z \in \rho(A)$. It means that $\sigma(A) \subset \mathbb{R}$. Since $A = A^*$ and therefore closed then the spectrum $\sigma(A)$ is closed as a complement of an open set (see part 1) of Theorem 1).

It remains to prove that $\sigma(A) \neq \emptyset$. Assume on the contrary that $\sigma(A) = \emptyset$. Then the resolvent R_z is an entire analytic function. Let us prove that $\|R_z\|$ is uniformly bounded with respect to $z \in \mathbb{C}$. Introduce the functional

$$T_z(y) := (R_z x, y), \quad \|x\| = 1, y \in H.$$

Then $T_z(y)$ is a linear functional on the Hilbert space H. Moreover, since R_z is bounded for any (fixed) $z \in \mathbb{C}$ then

$$|T_z(y)| \leq \|R_z x\| \|y\| \leq \|R_z\| \|y\| = C_z \|y\|.$$

Therefore $T_z(y)$ is continuous i.e. $\{T_z, z \in \mathbb{C}\}$ is a pointwise bounded family of continuous linear functionals. By Banach-Steinhaus theorem we may conclude that

$$\sup_{z \in \mathbb{C}} \|T_z\| = c_0 < \infty.$$

That's why we have

$$|T_z(y)| = |(R_z x, y)| \le c_0 \|y\|, \quad \|x\| = 1, z \in \mathbb{C}.$$

It implies that $\|R_z x\| \le c_0$ i.e. $\|R_z\| \le c_0$. By Liouville theorem we may conclude now that $R_z \equiv$ constant. But by J. von Neumann's spectral theorem

$$R_z = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} dE_\lambda,$$

where $\{E_\lambda\}$ is a spectral family of $A = A^*$. Due to the estimate

$$\|R_z\| \le \frac{1}{|\operatorname{Im} z|}$$

we may conclude that $\|R_z\| \to 0$ as $|\operatorname{Im} z| \to \infty$. Hence $R_z \equiv 0$. This contradiction finishes the proof. □

Exercise 22. [Weyl's criterion] Let $A = A^*$. Prove that $\lambda \in \sigma(A)$ if and only if there exists $x_n \in D(A), \|x_n\| = 1$ such that

$$\lim_{n \to \infty} \|(A - \lambda I)x_n\| = 0.$$

Definition. Let us assume that $A = \overline{A}$. The *point spectrum* $\sigma_p(A)$ of A is the set of eigenvalues of A i.e.

$$\sigma_p(A) = \{\lambda \in \sigma(A) : N(A - \lambda I) \neq \{0\}\}.$$

It means that $(A - \lambda I)^{-1}$ does not exist i.e. there exists a non-trivial $u \in D(A)$ such that $Au = \lambda u$. The complement $\sigma(A) \setminus \sigma_p(A)$ is the *continuous spectrum* $\sigma_c(A)$. The *discrete spectrum* is the set

$$\sigma_d(A) = \{\lambda \in \sigma_p(A) : \dim N(A - \lambda I) < \infty \,\text{and}\, \lambda \,\text{is isolated in}\, \sigma(A)\}.$$

The set $\sigma_{\text{ess}}(A) := \sigma(A) \setminus \sigma_d(A)$ is called the *essential spectrum* of A.

In the frame of this definition, the complex plane can be divided into regions according to

$$\mathbb{C} = \rho(A) \cup \sigma(A),$$
$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A)$$

and

$$\sigma(A) = \sigma_d(A) \cup \sigma_{\text{ess}}(A),$$

with all the unions being disjoint.

Remark. If $A = A^*$ then

1) $\lambda \in \sigma_c(A)$ means that $(A - \lambda I)^{-1}$ exists but is not bounded.

2) $\sigma_{\text{ess}}(A) = \overline{\sigma_c(A)} \cup \{\text{eigenvalues of infinite multiplicity and their accumulation points}\}$
$\cup \{\text{accumulation points of } \sigma_d(A)\}$.

Exercise 23. Let $A = A^*$ and $\lambda_1, \lambda_2 \in \sigma_p(A)$. Prove that if $\lambda_1 \neq \lambda_2$ then

$$N(A - \lambda_1 I) \perp N(A - \lambda_2 I).$$

Exercise 24. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis in H and let $\{s_j\}_{j=1}^{\infty} \subset \mathbb{C}$ be some sequence. Introduce the set

$$D = \left\{ x \in H : \sum_{j=1}^{\infty} |s_j|^2 |(x, e_j)|^2 < \infty \right\}.$$

Define

$$Ax = \sum_{j=1}^{\infty} s_j (x, e_j) e_j, \quad x \in D.$$

Prove that $A = \overline{A}$ and that $\sigma(A) = \overline{\{s_j : j = 1, 2, \ldots\}}$. Prove also that

$$(A - zI)^{-1} x = \sum_{j=1}^{\infty} \frac{1}{s_j - z} (x, e_j) e_j$$

for any $z \in \rho(A)$ and $x \in D$.

Exercise 25. Prove that the spectrum $\sigma(U)$ of a unitary operator U lies on the unit circle in $\mathbb{C}$.

Theorem 2. *Let $A = A^*$ and let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be its spectral family. Then*

1) $\mu \in \sigma(A)$ if and only if $E_{\mu+\varepsilon} - E_{\mu-\varepsilon} \neq 0$ for every $\varepsilon > 0$.

2) $\mu \in \sigma_p(A)$ if and only if $E_\mu - E_{\mu-0} \neq 0$. Here $E_{\mu-0} := \lim_{\varepsilon \to 0+} E_{\mu-\varepsilon}$ in the sense of strong operator topology.

Proof. 1) Suppose that $\mu \in \sigma(A)$ but there exists $\varepsilon > 0$ such that $E_{\mu+\varepsilon} - E_{\mu-\varepsilon} = 0$. Then by spectral theorem we obtain for any $x \in D(A)$ that

$$\begin{aligned} \|(A - \mu I)x\|^2 &= \int_{-\infty}^{\infty} (\lambda - \mu)^2 d(E_\lambda x, x) \geq \int_{|\lambda-\mu| \geq \varepsilon} (\lambda - \mu)^2 d(E_\lambda x, x) \\ &\geq \varepsilon^2 \int_{|\lambda-\mu| \geq \varepsilon} d(E_\lambda x, x) = \varepsilon^2 \left[\int_{-\infty}^{\mu-\varepsilon} + \int_{\mu+\varepsilon}^{\infty} \right] d(E_\lambda x, x) \\ &= \varepsilon^2 \left[(E_{\mu-\varepsilon} x, x) + \|x\|^2 - (E_{\mu+\varepsilon} x, x) \right] = \varepsilon^2 \|x\|^2. \end{aligned}$$

This inequality means (see part 2) of Theorem 1) that $\mu \notin \sigma(A)$ but $\mu \in \rho(A)$. This contradiction proves 1) in one direction. Conversely, if

$$P_n := E_{\mu+\frac{1}{n}} - E_{\mu-\frac{1}{n}} \neq 0$$

for all $n \in \mathbb{N}$ then there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in R(P_n)$ i.e. $x_n = P_n x_n$ i.e. $x_n \in D(A)$ and $\|x_n\| = 1$. For this sequence it is true that

$$\begin{aligned}\|(A-\mu I)x_n\|^2 &= \int_{-\infty}^{\infty} (\lambda-\mu)^2 d(E_\lambda P_n x_n, P_n x_n) = \int_{|\lambda-\mu|\le 1/n} (\lambda-\mu)^2 d(E_\lambda x_n, x_n) \\ &\le \frac{1}{n^2}\int_{-\infty}^{\infty} d(E_\lambda x_n, x_n) = \frac{1}{n^2}\|x_n\|^2 = \frac{1}{n^2} \to 0\end{aligned}$$

as $n \to \infty$. Hence, this sequence satisfies Weyl's criterion (see Exercise 22) and therefore $\mu \in \sigma(A)$.

2) Suppose $\mu \in \mathbb{R}$ is an eigenvalue of A. Then there is $x_0 \in D(A), x_0 \neq 0$ such that

$$0 = \|(A-\mu I)x_0\|^2 = \int_{-\infty}^{\infty} (\lambda-\mu)^2 d(E_\lambda x_0, x_0).$$

In particular, for all $n \in \mathbb{N}$ and large enough $\varepsilon > 0$ we have that

$$\begin{aligned}0 &= \int_{\mu+\varepsilon}^{n} (\lambda-\mu)^2 d(E_\lambda x_0, x_0) \ge \varepsilon^2 \int_{\mu+\varepsilon}^{n} d(E_\lambda x_0, x_0) = \varepsilon^2((E_n - E_{\mu+\varepsilon})x_0, x_0) \\ &= \varepsilon^2 \|(E_n - E_{\mu+\varepsilon})x_0\|^2.\end{aligned}$$

Thus we may conclude that

$$0 = E_n x_0 - E_{\mu+\varepsilon} x_0.$$

Similarly we can get that

$$0 = E_{-n} x_0 - E_{\mu-\varepsilon} x_0.$$

Letting $n \to \infty$ and $\varepsilon \to 0$ we obtain

$$x_0 = E_\mu x_0, \quad 0 = E_{\mu-0} x_0.$$

Hence

$$x_0 = (E_\mu - E_{\mu-0})x_0$$

and therefore

$$E_\mu - E_{\mu-0} \neq 0.$$

Conversely, define the projector

$$P := E_\mu - E_{\mu-0}.$$

If $P \neq 0$ then there exists $y \in H, y \neq 0$ such that $y = Py$ (e.g. any $y \in R(P) \neq \{0\}$ will do). For $\lambda > \mu$ it follows that

$$E_\lambda y = E_\lambda P y = E_\lambda E_\mu y - E_\lambda E_{\mu-0} y = Py = y.$$

For $\lambda < \mu$ we have that

$$E_\lambda y = E_\lambda E_\mu y - E_\lambda E_{\mu-0} y = E_\lambda y - E_\lambda y = 0.$$

Hence

$$\|(A-\mu I)y\|^2 = \int_{-\infty}^{\infty} (\lambda-\mu)^2 d(E_\lambda y, y) = \int_{\mu}^{\infty} (\lambda-\mu)^2 d_\lambda(y,y) = 0.$$

That's why $Ay = \mu y$ and $y \in D(A), y \neq 0$ i.e. μ is an eigenvalue of A or $\mu \in \sigma_p(A)$. □

Remark. The statements of Theorem 2 can be reformulated as

1) $\mu \in \sigma_p(A)$ if and only if $E_\mu - E_{\mu-0} \neq 0$.

2) $\mu \in \sigma_c(A)$ if and only if $E_\mu - E_{\mu-0} = 0$.

Definition. Let H and H_1 be two Hilbert spaces. A bounded linear operator $K : H \to H_1$ is called *compact* or *completely continuous* if it maps bounded sets in H into *precompact* sets in H_1 i.e. for every bounded sequence $\{x_n\}_{n=1}^{\infty} \subset H$ the sequence $\{Kx_n\}_{n=1}^{\infty} \subset H_1$ contains a convergent subsequence.

If $K : H \to H_1$ is compact then the following statements hold.

1) K maps every weakly convergent sequence in H into a norm convergent sequence in H_1.

2) If $H = H_1$ is separable then every compact operator is a norm limit of a sequence of operators of *finite rank* (i.e. operators with finite dimensional ranges).

3) The norm limit of a sequence of compact operators is compact.

Let us prove 2). Let K be a compact operator. Since H is separable it has an orthonormal basis $\{e_j\}_{j=1}^{\infty}$. Consider for any $n = 1, 2, \ldots$ the projector

$$P_n x := \sum_{j=1}^{n} (x, e_j) e_j, \quad x \in H.$$

Then $P_n \leq P_{n+1}$ and $\|(I-P_n)x\| \to 0$ as $n \to \infty$. Define

$$d_n := \sup_{\|x\|=1} \|K(I-P_n)x\| \equiv \|K(I-P_n)\|.$$

Since $R(I-P_n) \supset R(I-P_{n+1})$ (see Proposition 2 in Section 3) then $\{d_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence of positive numbers. Hence the limit

$$\lim_{n\to\infty} d_n := d \geq 0$$

exists. Let us choose $y_n \in R(I-P_n), \|y_n\| = 1$ such that

$$\|K(I-P_n)y_n\| = \|Ky_n\| \geq \frac{d}{2}.$$

Then

$$|(y_n, x)| = |((I - P_n)y_n, x)| = |(y_n, (I - P_n)x)| \leq \|y_n\| \, \|(I - P_n)x\| \to 0, \quad n \to \infty$$

for any $x \in H$. It means that $y_n \xrightarrow{w} 0$. Compactness of K implies that $Ky_n \to 0$. Thus $d = 0$. That's why

$$d_n = \|K - KP_n\| \to 0.$$

Since P_n is of finite rank then so is KP_n i.e. K is a norm limit of finite rank operators.

Lemma. *Suppose $A = A^*$ is compact. Then at least one of the two numbers $\pm \|A\|$ is an eigenvalue of A.*

Proof. Since

$$\|A\| = \sup_{\|x\|=1} |(Ax, x)|$$

then there exists a sequence x_n with $\|x_n\| = 1$ such that

$$\|A\| = \lim_{n\to\infty} |(Ax_n, x_n)|.$$

Actually, we can assume that $\lim_{n\to\infty}(Ax_n, x_n)$ exists and equals, say, a. Otherwise we would take a subsequence of $\{x_n\}$. Since $A = A^*$ then a is real and $\|A\| = |a|$. Due to the fact that any bounded set of the Hilbert space is weakly compact (unit ball in our case) we can choose a subsequence of $\{x_n\}$, say, $\{x_{k_n}\}$ which converges weakly i.e. $x_{k_n} \xrightarrow{w} x$. Compactness of A implies that $Ax_{k_n} \to y$. Next we observe that

$$\begin{aligned} \|Ax_{k_n} - ax_{k_n}\|^2 &= \|Ax_{k_n}\|^2 - 2a(Ax_{k_n}, x_{k_n}) + a^2 \leq \|A\|^2 - 2a(Ax_{k_n}, x_{k_n}) + a^2 \\ &= 2a^2 - 2a(Ax_{k_n}, x_{k_n}) \to 2a^2 - 2a^2 = 0, \end{aligned}$$

as $n \to \infty$. Hence

$$\begin{cases} Ax_{k_n} - ax_{k_n} \to 0 \\ Ax_{k_n} \to y \\ x_{k_n} \xrightarrow{w} x \end{cases} \Rightarrow \quad \begin{cases} x_{k_n} \to x \\ Ax = ax. \end{cases}$$

Since $\|x_{k_n}\| = 1$ then $\|x\| = 1$ also. Hence $x \neq 0$ and a is an eigenvalue of A. □

Theorem 3 (Riesz-Schauder)**.** *Suppose $A = A^*$ is compact. Then*

1) *A has a sequence of real eigenvalues $\lambda_j \neq 0$ which can be enumerated in such a way that*

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_j| \geq \cdots.$$

2) *If there are infinitely many eigenvalues then $\lim_{j\to\infty} \lambda_j = 0$ and 0 is the only accumulation point of $\{\lambda_j\}$.*

3) *The multiplicity of λ_j is finite.*

4) If e_j is the normalized eigenvector for λ_j then $\{e_j\}_{j=1}^{\infty}$ is an orthonormal system and

$$Ax = \sum_{j=1}^{\infty} \lambda_j (x, e_j) e_j = \sum_{j=1}^{\infty} (Ax, e_j) e_j, \quad x \in H.$$

It means that $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis on $R(A)$.

5) $\sigma(A) = \{0, \lambda_1, \lambda_2, \dots, \lambda_j, \dots\}$ while 0 is not necessarily an eigenvalue of A.

Proof. Lemma gives the existence of an eigenvalue $\lambda_1 \in \mathbb{R}$ with $|\lambda_1| = \|A\|$ and a normalized eigenvector e_1. Introduce $H_1 = e_1^{\perp}$. Then H_1 is a closed subspace of H and A maps H_1 into itself. Indeed,

$$(Ax, e_1) = (x, Ae_1) = (x, \lambda_1 e_1) = \lambda_1 (x, e_1) = 0$$

for any $x \in H_1$. The restriction of the inner product of H to H_1 makes H_1 a Hilbert space (since H_1 is closed) and the restriction of A to H_1, denoted by $A_1 = A|_{H_1}$, is again a self-adjoint compact operator which is mapping in H_1. Clearly, its norm is bounded by the norm of A i.e. $\|A_1\| \le \|A\|$. Applying Lemma to A_1 on H_1 we get an eigenvalue λ_2 with $|\lambda_2| = \|A_1\|$ and a normalized eigenvector e_2 with $e_2 \perp e_1$. It is clear that $|\lambda_2| \le |\lambda_1|$. Next introduce the closed subspace $H_2 = (\operatorname{span}\{e_1, e_2\})^{\perp}$. Again, A leaves H_2 invariant and thus $A_2 := A_1|_{H_2} = A|_{H_2}$ is a self-adjoint compact operator in H_2. Applying Lemma to A_2 on H_2 we obtain λ_3 with $|\lambda_3| = \|A_2\|$ and a normalized eigenvector e_3 with $e_3 \perp e_2$ and $e_3 \perp e_1$. This process in the infinite dimensional Hilbert space leads us to the sequence $\{\lambda_j\}_{j=1}^{\infty}$ such that $|\lambda_{j+1}| \le |\lambda_j|$ and corresponding normalized eigenvectors. Since $|\lambda_j| > 0$ and monotone decreasing then there is a limit

$$\lim_{j \to \infty} |\lambda_j| = r.$$

Clearly $r \ge 0$. Let us prove that $r = 0$. If $r > 0$ then $|\lambda_j| \ge r > 0$ for each $j = 1, 2, \dots$ or

$$\frac{1}{|\lambda_j|} \le \frac{1}{r} < \infty.$$

Hence the sequence of vectors

$$y_j := \frac{e_j}{\lambda_j}$$

is bounded and therefore there is a weakly convergent subsequence $y_{j_k} \xrightarrow{w} y$. Compactness of A implies the strong convergence of $Ay_{j_k} \equiv e_{j_k}$. But $\|e_{j_k} - e_{j_m}\| = \sqrt{2}$ for $k \ne m$. This contradiction proves 1) and 2).

Exercise 26. Prove that if H is an infinite dimensional Hilbert space then the identical operator I is not compact and inverse of a compact operator (if it exists) is not bounded.

Exercise 27. Prove part 3) of Theorem 3.

Consider now the projector

$$P_n x := \sum_{j=1}^{n} (x, e_j) e_j, \quad x \in H.$$

Then $I - P_n$ is a projector onto $(\operatorname{span}\{e_1, \dots, e_n\})^{\perp} \equiv H_n$ and hence

$$\|A(I - P_n)x\| \le \|A\|_{H_n} \|(I - P_n)x\| \le |\lambda_{n+1}| \, \|x\| \to 0$$

as $n \to \infty$. Since

$$AP_n x = \sum_{j=1}^{n} (x, e_j) A e_j = \sum_{j=1}^{n} \lambda_j (x, e_j) e_j$$

and

$$\|A(I - P_n)x\| = \|Ax - AP_n x\| \to 0, \quad n \to \infty$$

then

$$Ax = \sum_{j=1}^{\infty} \lambda_j (x, e_j) e_j$$

and part 4) follows. Finally, Exercise 24 gives immediately that

$$\sigma(A) = \{0, \lambda_1, \lambda_2, \dots, \lambda_j, \dots\}.$$

This finishes the proof. □

Corollary (Hilbert-Schmidt theorem)**.** *The orthonormal system $\{e_j\}_{j=1}^{\infty}$ of eigenvectors of a compact self-adjoint operator A in a Hilbert space H is an orthonormal basis if and only if $N(A) = \{0\}$.*

Proof. Recall from Exercise 13 that

$$H = N(A^*) \oplus \overline{R(A)} = N(A) \oplus \overline{R(A)}.$$

If $N(A) = \{0\}$ then $H = \overline{R(A)}$. It means that for any $x \in H$ and any $\varepsilon > 0$ there exists $y_\varepsilon \in R(A)$ such that

$$\|x - y_\varepsilon\| < \varepsilon/2.$$

But by Riesz-Schauder theorem

$$y_\varepsilon = Ax_\varepsilon = \sum_{j=1}^{\infty} \lambda_j (x_\varepsilon, e_j) e_j.$$

Hence

$$\|x - y_\varepsilon\| = \left\| x - \sum_{j=1}^{\infty} \lambda_j (x_\varepsilon, e_j) e_j \right\| < \varepsilon/2.$$

Making use of the Theorem of Pythagoras, Bessel's inequality and Exercise 8 yields

$$\begin{aligned}
\left\| x - \sum_{j=1}^{n} (x, e_j) e_j \right\| &\le \left\| x - \sum_{j=1}^{n} \lambda_j (x_\varepsilon, e_j) e_j \right\| = \left\| x - \sum_{j=1}^{\infty} \lambda_j (x_\varepsilon, e_j) e_j + \sum_{j=n+1}^{\infty} \lambda_j (x_\varepsilon, e_j) e_j \right\| \\
&< \varepsilon/2 + \left\| \sum_{j=n+1}^{\infty} \lambda_j (x_\varepsilon, e_j) e_j \right\| \\
&\le \varepsilon/2 + \left(\sum_{j=n+1}^{\infty} |\lambda_j|^2 |(x_\varepsilon, e_j)|^2 \right)^{1/2} \\
&\le \varepsilon/2 + |\lambda_{n+1}| \left(\sum_{j=n+1}^{\infty} |(x_\varepsilon, e_j)|^2 \right)^{1/2} \\
&\le \varepsilon/2 + |\lambda_{n+1}| \, \|x_\varepsilon\| < \varepsilon
\end{aligned}$$

for n large enough. It means that $\{e_j\}_{j=1}^{\infty}$ is a basis in H, and moreover, it is an orthonormal basis.

Conversely, if $\{e_j\}_{j=1}^{\infty}$ is complete in H then $\overline{R(A)} = H$ (Riesz-Schauder) and therefore $N(A) = \{0\}$. □

Remark. The condition $N(A) = \{0\}$ means that A^{-1} exists and H must be separable in this case.

Theorem 4 (Lemma of Riesz). *If A is a compact operator on H and $\mu \in \mathbb{C}$ then $R(I - \mu A)$ is closed in H.*

Proof. If $\mu = 0$ then $R(I - \mu A) = H$. If $\mu \neq 0$ then we assume without loss of generality that $\mu = 1$. Let $f \in \overline{R(I - A)}, f \neq 0$. Then there exists a sequence $\{g_n\} \subset H$ such that

$$f = \lim_{n \to \infty} (I - A) g_n.$$

We will prove that $f \in R(I - A)$ i.e. there exists $g \in H$ such that $f = (I - A)g$. Since $f \neq 0$ then by the decomposition $H = N(I - A) \oplus N(I - A)^{\perp}$ we can assume that $g_n \in N(I - A)^{\perp}$ and $g_n \neq 0$ for all $n \in \mathbb{N}$.

Suppose that g_n is bounded. Then there is a subsequence $\{g_{k_n}\}$ such that

$$g_{k_n} \xrightarrow{w} g.$$

Compactness of A implies that

$$A g_{k_n} \to h = A g.$$

Next,

$$g_{k_n} = (I - A) g_{k_n} + A g_{k_n} \to f + h.$$

Hence $g = f + Ag$ i.e. $f = (I - A)g$.

Suppose that g_n is not bounded. Then we can assume without loss of generality that $\|g_n\| \to \infty$. Introduce a new sequence

$$u_n := \frac{g_n}{\|g_n\|}.$$

Since $\|u_n\| = 1$ then there exists a subsequence $u_{k_n} \xrightarrow{w} u$. Compactness of A gives $Au_{k_n} \to Au$. Since $(I - A)g_n \to f$ then

$$(I - A)u_{k_n} = \frac{1}{\|g_{k_n}\|}(I - A)g_{k_n} \to 0.$$

It means again that

$$u_{k_n} = (I - A)u_{k_n} + Au_{k_n} \to Au$$

and $u = Au$ i.e. $u \in N(I - A)$. But $g_n \in N(I - A)^\perp$. Hence $u_{k_n} \in N(I - A)^\perp$ and further $u \in N(I - A)^\perp$ because $N(I - A)^\perp$ is closed. Since $\|u_{k_n}\| = 1$ then $\|u\| = 1$. Therefore $u \neq 0$ while

$$u \in N(I - A) \cap N(I - A)^\perp.$$

This contradiction shows that unbounded g_n cannot occur. □

Theorem 5 (Fredholm alternative)**.** *Suppose $A = A^*$ is compact. For given $g \in H$ either the equation*

$$(I - \mu A)f = g$$

has the unique solution ($\mu^{-1} \notin \sigma(A)$) and in this case $f = (I - \mu A)^{-1}g$ or $\mu^{-1} \in \sigma(A)$ and this equation has a solution if and only if $g \in R(I - \mu A)$ i.e. $g \perp N(I - \mu A)$. In this case the general solution of the equation is of the form $f = f_0 + u$, where f_0 is a particular solution and $u \in N(I - \mu A)$ (u is the general solution of the corresponding homogeneous equation) and the set of all solutions is a finite dimensional affine subspace of H.

Proof. Lemma of Riesz (Theorem 4) gives

$$R(I - \mu A) = N(I - \overline{\mu}A)^\perp.$$

If $\mu^{-1} \notin \sigma(A)$ then $(\overline{\mu})^{-1} \notin \sigma(A)$ also. Thus

$$R(I - \mu A) = N(I - \overline{\mu}A)^\perp = \{0\}^\perp = H.$$

Since $A = A^*$ this means that $(I - \mu A)^{-1}$ exists and the unique solution is $f = (I - \mu A)^{-1}g$.

If $\mu^{-1} \in \sigma(A)$ then $R(I - \mu A)$ is a proper subspace of H and the equation $(I - \mu A)f = g$ has a solution if and only if $g \in R(I - \mu A)$. Since the equation is linear then any solution is of the form

$$f = f_0 + u, \quad u \in N(I - \mu A)$$

and the dimension of $N(I - \mu A)$ is finite. □

Exercise 28. Let $A = A^*$ be compact. Prove that $\sigma_p(A) = \sigma_d(A) = \sigma(A) \setminus \{0\}$ and $0 \in \sigma_{\text{ess}}(A)$.

Exercise 29. Consider the Hilbert space $H = l^2(\mathbb{C})$ and

$$A(x_1, x_2, \ldots, x_n, \ldots) = (0, x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots)$$

for $(x_1, x_2, \ldots, x_n, \ldots) \in l^2(\mathbb{C})$. Show that A is compact and has no eigenvalues (even more, $\sigma(A) = \emptyset$) and is not self-adjoint.

Exercise 30. Consider the Hilbert space $H = L^2(\mathbb{R})$ and

$$(Af)(t) = tf(t).$$

Show that the equation $Af = f$ has no non-trivial solutions and that $(I - A)^{-1}$ does not exist. It means that the Fredholm alternative does not hold for non-compact but self-adjoint operator.

Exercise 31. Let $H = L^2(\mathbb{R}^n)$ and let

$$Af(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy,$$

where $K(x, y) \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is such that $K(x, y) = \overline{K(y, x)}$. Prove that $A = A^*$ and that A is compact.

Theorem 6 (Weyl)**.** *If $A = A^*$ then $\lambda \in \sigma_{ess}(A)$ if and only if there exists an orthonormal system $\{x_n\}_{n=1}^{\infty}$ such that*

$$\|(A - \lambda I)x_n\| \to 0$$

as $n \to \infty$.

Proof. We will provide only a partial proof. Suppose that $\lambda \in \sigma_{\text{ess}}(A)$. If λ is an eigenvalue of infinite multiplicity then there is an infinite orthonormal system of eigenvectors $\{x_n\}_{n=1}^{\infty}$ because $\dim(E_\lambda - E_{\lambda-0})H = \infty$ in this case. Since $(A - \lambda I)x_n \equiv 0$ it is clear that

$$(A - \lambda I)x_n \to 0.$$

Next, suppose that λ is an accumulation point of $\sigma(A)$. It means that $\lambda \in \sigma(A)$ and

$$\lambda = \lim_{n\to\infty} \lambda_n,$$

where $\lambda_n \neq \lambda_m, n \neq m$ and $\lambda_n \in \sigma(A)$. Hence for each $n = 1, 2, \ldots$ we have that

$$E_{\lambda_n+\varepsilon} - E_{\lambda_n-\varepsilon} \neq 0$$

for all $\varepsilon > 0$. Therefore there exists a sequence $r_n \to 0$ such that

$$E_{\lambda_n+r_n} - E_{\lambda_n-r_n} \neq 0.$$

That's why we can find a normalized vector $x_n \in R(E_{\lambda_n+r_n} - E_{\lambda_n-r_n})$. Since $\lambda_n \neq \lambda_m$ for $n \neq m$ we can find $\{x_n\}_{n=1}^{\infty}$ as an orthonormal system. By spectral theorem we have

$$\begin{aligned}
\|(A-\lambda I)x_n\|^2 &= \int_{-\infty}^{\infty}(\lambda-\mu)^2 d(E_\mu x_n, x_n) \\
&= \int_{-\infty}^{\infty}(\lambda-\mu)^2 d(E_\mu(E_{\lambda_n+r_n} - E_{\lambda_n-r_n})x_n, x_n) \\
&= \int_{\lambda_n-r_n}^{\lambda_n+r_n}(\lambda-\mu)^2 d(E_\mu x_n, x_n) \\
&\leq \max_{\lambda_n-r_n\leq\mu\leq\lambda_n+r_n}(\lambda-\mu)^2 \int_{-\infty}^{\infty} d(E_\mu x_n, x_n) \\
&= \max_{\lambda_n-r_n\leq\mu\leq\lambda_n+r_n}(\lambda-\mu)^2 \to 0, \quad n \to \infty.
\end{aligned}$$

□

Theorem 7 (Weyl)**.** *Let A and B be two self-adjoint operators in a Hilbert space. If there is $z \in \rho(A) \cap \rho(B)$ such that*

$$T := (A - zI)^{-1} - (B - zI)^{-1}$$

is a compact operator then $\sigma_{ess}(A) = \sigma_{ess}(B)$.

Proof. We show first that $\sigma_{\text{ess}}(A) \subset \sigma_{\text{ess}}(B)$. Take any $\lambda \in \sigma_{\text{ess}}(A)$. Then there is an orthonormal system $\{x_n\}_{n=1}^{\infty}$ such that

$$\|(A - \lambda I)x_n\| \to 0, \quad n \to \infty.$$

Define the sequence y_n as

$$y_n := (A - zI)x_n \equiv (A - \lambda I)x_n + (\lambda - z)x_n.$$

Due to Bessel's inequality any orthonormal system in the Hilbert space converges weakly to 0. Hence $y_n \xrightarrow{w} 0$. We also have

$$\|y_n\| \geq |\lambda - z| \, \|x_n\| - \|(A - \lambda I)x_n\| = |\lambda - z| - \|(A - \lambda I)x_n\| > \frac{|\lambda - z|}{2} > 0$$

for all $n \geq n_0 >> 1$. Next we take the identity

$$\left[(B - zI)^{-1} - (\lambda - z)^{-1}\right] y_n = -Ty_n - (\lambda - z)^{-1}(A - \lambda I)x_n.$$

Since T is compact and $y_n \xrightarrow{w} 0$ we deduce that

$$\left[(B - zI)^{-1} - (\lambda - z)^{-1}\right] y_n \to 0.$$

Introduce

$$z_n := (B - zI)^{-1}y_n.$$

Then

$$z_n - (\lambda - z)^{-1} y_n \to 0$$

or

$$y_n + (z - \lambda) z_n \to 0.$$

This fact and $\|y_n\| > \frac{|\lambda - z|}{2}$ imply that $\|z_n\| \geq \frac{|\lambda - z|}{3}$ for all $n \geq n_0 >> 1$. But

$$(B - \lambda I) z_n \equiv (B - zI) z_n + (z - \lambda) z_n = y_n + (z - \lambda) z_n \to 0.$$

Due to $\|z_n\| \geq \frac{|\lambda - z|}{3} > 0$ the sequence $\{z_n\}_{n=1}^{\infty}$ can be chosen as an orthonormal system. Thus $\lambda \in \sigma_{\text{ess}}(B)$. This proves that $\sigma_{\text{ess}}(A) \subset \sigma_{\text{ess}}(B)$. Finally, since $-T$ is compact too we can interchange the roles of A and B and obtain the opposite embedding. $\square$

5 Quadratic forms. Friedrichs extension.

Definition. Let D be a linear subspace of a Hilbert space H. A function $Q : D \times D \to \mathbb{C}$ is called a *quadratic form* if

1) $Q(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 Q(x_1, y) + \alpha_2 Q(x_2, y)$
2) $Q(x, \beta_1 y_1 + \beta_2 y_2) = \overline{\beta_1} Q(x, y_1) + \overline{\beta_2} Q(x, y_2)$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ and $x_1, x_2, x, y_1, y_2, y \in D$. The space $D(Q) := D$ is called the *domain* of Q. We say that Q is

a) *densely defined* if $\overline{D(Q)} = H$.

b) *symmetric* if $Q(x, y) = \overline{Q(y, x)}$.

c) *semibounded from below* if there exists $\lambda \in \mathbb{R}$ such that $Q(x, x) \geq -\lambda \|x\|^2$ for all $x \in D(Q)$.

d) *closed* (and semibounded) if $D(Q)$ is complete with respect to the norm

$$\|x\|_Q := \sqrt{Q(x, x) + (\lambda + 1) \|x\|^2}.$$

e) *bounded* (continuous) if there exists $M > 0$ such that

$$|Q(x, y)| \leq M \|x\| \|y\|$$

for all $x, y \in D(Q)$.

Exercise 32. Prove that $\|\cdot\|_Q$ is a norm and that

$$(x, y)_Q := Q(x, y) + (\lambda + 1)(x, y)$$

is an inner product.

Theorem 1. *Let Q be a densely defined, closed, semibounded and symmetric quadratic form in a Hilbert space H such that*

$$Q(x, x) \geq -\lambda \|x\|^2, \quad x \in D(Q).$$

Then there exists a unique self-adjoint operator A which is semi-bounded from below i.e.

$$(Ax, x) \geq -\lambda \|x\|^2, \quad x \in D(A).$$

Moreover, this operator A defines the quadratic form Q as

$$Q(x, y) = (Ax, y), \quad x \in D(A), y \in D(Q)$$

and $D(A) \subset D(Q)$.

Proof. Let us introduce an inner product on $D(Q)$ by

$$(x, y)_Q := Q(x, y) + (\lambda + 1)(x, y), \quad x, y \in D(Q)$$

(see Exercise 32). Since Q is closed then $D(Q) = \overline{D(Q)}$ is a closed subspace of H with respect to the norm $\|\cdot\|_Q$. It means that $D(Q)$ with this inner product defines a new Hilbert space H_Q. It is clear also that

$$\|x\|_Q \geq \|x\|$$

for all $x \in H_Q$. Thus, for fixed $x \in H$,

$$L(y) := (y, x), \quad y \in H_Q$$

defines a continuous (bounded) linear functional on the Hilbert space H_Q. Applying the Riesz-Frechet theorem to H_Q we obtain an element $x^* \in H_Q$ ($x^* \in D(Q)$) such that

$$(y, x) \equiv L(y) = (y, x^*)_Q.$$

It is clear that the map

$$H \ni x \mapsto x^* \in H_Q$$

defines a linear operator J such that

$$J : H \to H_Q, \quad Jx = x^*.$$

Hence

$$(y, x) = (y, Jx)_Q, \quad x \in H, y \in H_Q.$$

Next we prove that J is self-adjoint and that it has an inverse operator J^{-1}. For any $x, y \in H$ we have

$$(Jy, x) = (Jy, Jx)_Q = \overline{(Jx, Jy)_Q} = \overline{(Jx, y)} = (y, Jx).$$

Hence $J = J^*$. It is bounded due to Hellinger-Toeplitz theorem (Exercise 9). Suppose that $Jx = 0$. Then

$$(y, x) = (y, Jx)_Q = 0$$

for any $y \in D(Q)$. Since $\overline{D(Q)} = H$ then the last equality implies that $x = 0$ and therefore $N(J) = \{0\}$ and J^{-1} exists. Moreover,

$$H = N(J) \oplus \overline{R(J^*)} = \overline{R(J)}$$

and $R(J) \subset H_Q$. Now we can define a linear operator A on the domain $D(A) \equiv R(J)$ as

$$Ax := J^{-1}x - (\lambda + 1)x, \quad \lambda \in \mathbb{R}.$$

It is clear that A is densely defined and $A = A^*$ (J^{-1} is self-adjoint since J is). If now $x \in D(A)$ and $y \in D(Q) \equiv H_Q$ then

$$Q(x, y) = (x, y)_Q - (\lambda + 1)(x, y) = (J^{-1}x, y) - (\lambda + 1)(x, y) = (Ax, y).$$

The semi-boundedness of A from below follows from that of Q. It remains to prove that this representation for A is unique. Assume that we have two such representations, A_1 and A_2. Then for every $x \in D(A_1) \cap D(A_2)$ and $y \in D(Q)$ we have that

$$Q(x, y) = (A_1 x, y) = (A_2 x, y).$$

It follows that

$$((A_1 - A_2)x, y) = 0.$$

Since $\overline{D(Q)} = H$ then we must have $A_1 x = A_2 x$. This finishes the proof. □

Corollary. *Under the same assumptions as in Theorem 1, there exists $\sqrt{A + \lambda I}$ which is self-adjoint on $D(\sqrt{A + \lambda I}) \equiv D(Q) = H_Q$. Moreover,*

$$Q(x, y) + \lambda(x, y) = (\sqrt{A + \lambda I}x, \sqrt{A + \lambda I}y)$$

for all $x, y \in D(Q)$.

Proof. Since $A + \lambda I$ is self-adjoint and non-negative there exists a spectral family $\{E_\mu\}_{\mu=0}^{\infty}$ such that

$$A + \lambda I = \int_0^\infty \mu dE_\mu.$$

That's why we can define the operator

$$\sqrt{A + \lambda I} := \int_0^\infty \sqrt{\mu} dE_\mu$$

which is also self-adjoint and non-negative. Then for any $x \in D(A)$ and $y \in D(Q)$ we have that

$$Q(x, y) + \lambda(x, y) = ((A + \lambda I)x, y) = (\sqrt{A + \lambda I}x, \left(\sqrt{A + \lambda I}\right)^* y).$$

This means that $x \in D(\sqrt{A + \lambda I})$ and $y \in D(\left(\sqrt{A + \lambda I}\right)^*)$. But $\sqrt{A + \lambda I}$ is self-adjoint and, therefore,

$$D(\sqrt{A + \lambda I}) = D(\left(\sqrt{A + \lambda I}\right)^*) = D(Q) \equiv H_Q.$$

□

Theorem 2 (Friedrichs extension)**.** *Let A be a non-negative, symmetric linear operator in a Hilbert space H. Then there exists a self-adjoint extension A_F of A which is the smallest among all non-negative self-adjoint extensions of A in the sense that its corresponding quadratic form has the smallest domain. This extension A_F is called the* Friedrichs extension *of A.*

Proof. Let A be a non-negative, symmetric operator with domain $D(A)$ dense in H, $\overline{D(A)} = H$. Its associated quadratic form

$$Q(x, y) := (Ax, y), \quad x, y \in D(Q) \equiv D(A)$$

is densely defined, non-negative and symmetric. Let us define a new inner product

$$(x, y)_Q = Q(x, y) + (x, y), \quad x, y \in D(Q).$$

Then $D(Q)$ becomes an inner product space. This inner product space has a completion H_Q with respect to the norm

$$\|x\|_Q := \sqrt{Q(x, x) + \|x\|^2}.$$

Moreover, the quadratic form $Q(x, y)$ has an extension $Q_1(x, y)$ to this Hilbert space H_Q defined by

$$Q_1(x, y) = \lim_{n \to \infty} Q(x_n, y_n)$$

whenever $x \overset{H_Q}{=} \lim_{n\to\infty} x_n, y \overset{H_Q}{=} \lim_{n\to\infty} y_n, x_n, y_n \in D(Q)$ and these limits exist. The quadratic form Q_1 is densely defined, closed, non-negative and symmetric. That's why Theorem 1, applied to Q_1, gives a unique and non-negative, self-adjoint operator A_F such that

$$Q_1(x, y) = (A_F x, y), \quad x \in D(A_F) \subset H_Q, y \in D(Q_1) \equiv H_Q.$$

Since for $x, y \in D(A)$ one has

$$(Ax, y) = Q(x, y) = Q_1(x, y) = (A_F x, y)$$

then A_F is a self-adjoint extension of A.

It remains to prove that A_F is the smallest non-negative self-adjoint extension of A. Suppose that $B \geq 0, B = B^*$ is such that $A \subset B$. The associated quadratic form $Q_B(x, y) := (Bx, y)$ is an extension of $Q \equiv Q_A$. Hence

$$\overline{Q_B} \supset \overline{Q} = Q_1.$$

This finishes the proof. $\square$

6 Elliptic differential operators

Let Ω be a domain in $\mathbb{R}^n$ i.e. an open and connected set. Introduce the following notation:

1) $x = (x_1, \ldots, x_n) \in \Omega$
2) $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$
3) $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a *multi-index* i.e. $\alpha_j \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$.
    a) $|\alpha| = \alpha_1 + \cdots + \alpha_n$
    b) $\alpha \geq \beta$ if $\alpha_j \geq \beta_j$ for all $j = 1, 2, \ldots, n$.
    c) $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$
    d) $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ if $\alpha \geq \beta$
    e) $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $0^0 = 1$
    f) $\alpha! = \alpha_1! \cdots \alpha_n!$ with $0! = 1$
4) $D_j = \frac{1}{i}\partial_j = \frac{1}{i}\frac{\partial}{\partial x_j} = -i\partial_j$ and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \equiv (-i)^{|\alpha|}\partial^\alpha$

Definition. An *elliptic partial differential operator* $A(x, D)$ of order m on Ω is an operator of the form

$$A(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

where $a_\alpha(x) \in C^\infty(\Omega)$ and whose *principal symbol*

$$a(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha, \quad \xi \in \mathbb{R}^n$$

is invertible for all $x \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, that is, $a(x, \xi) \neq 0$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.

Assumption 1. We assume that $a_\alpha(x)$ are real for $|\alpha| = m$.

Under Assumption 1 either $a(x, \xi) > 0$ or $a(x, \xi) < 0$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. Without loss of generality we assume that $a(x, \xi) > 0$. Assumption 1 implies also that m is even and for any compact set $K \subset \Omega$ there exists $C_K > 0$ such that

$$a(x, \xi) \geq C_K |\xi|^m, \quad x \in \Omega, \xi \in \mathbb{R}^n.$$

Assumption 2. We assume that $A(x, D)$ is *formally self-adjoint* i.e.

$$A(x, D) = A^*(x, D) := \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha(\overline{a_\alpha(x)} \cdot).$$

Exercise 33. Prove that $A(x,D) = A^*(x,D)$ if and only if

$$a_\alpha(x) = \sum_{\substack{\alpha\le\beta\\|\beta|\le m}} (-1)^{|\beta|} C_\beta^\alpha D^{\beta-\alpha}\overline{a_\beta(x)},$$

where

$$C_\beta^\alpha = \frac{\beta!}{\alpha!(\beta-\alpha)!}.$$

Hint: Make use of the *generalized Leibniz formula*

$$D^\alpha(fg) = \sum_{\beta\le\alpha} C_\alpha^\beta D^{\alpha-\beta} f D^\beta g.$$

Assumption 3. We assume that $A(x,D)$ has a divergence form

$$A(x,D) \equiv \sum_{|\alpha|=|\beta|\le m/2} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)D^\beta),$$

where $a_{\alpha\beta} = a_{\beta\alpha}$ and real for all α and β. We assume also the *ellipticity condition*

$$\sum_{|\alpha|=|\beta|=m/2} a_{\alpha\beta}(x)\xi^\alpha\xi^\beta \ge \nu \sum_{|\alpha|=m/2} |\xi^\alpha|^2 = \nu \sum_{|\alpha|=m/2} \xi^{2\alpha},$$

where $\nu > 0$ is called the *constant of ellipticity*. Such operator is called *uniformly elliptic*.

Exercise 34. Prove that

$$\sum_{|\alpha|=m/2} \xi^{2\alpha} \asymp |\xi|^m$$

i.e.

$$c|\xi|^m \le \sum_{|\alpha|=m/2} \xi^{2\alpha} \le C|\xi|^m,$$

where c and C are some constants.

Example 6.1. Let us consider

$$A(x,D) = \sum_{j=1}^n D_j^2 = -\Delta, \quad x \in \Omega \subset \mathbb{R}^n$$

in $H = L^2(\Omega)$ and prove that $A \subset A^*$ with

$$D(A) = C_0^\infty(\Omega) = \left\{ f \in C^\infty(\Omega) : \operatorname{supp} f = \overline{\{x : f(x) \ne 0\}} \text{ is compact in } \Omega \right\}.$$

Let $u, v \in C_0^\infty(\Omega)$. Then

$$\begin{aligned}
(Au,v)_{L^2} &= \int_\Omega \left(\sum_{j=1}^n D_j^2 u\right)\overline{v}dx = -\sum_{j=1}^n \int_\Omega \left(\partial_j^2 u\right)\overline{v}dx \\
&= -\sum_{j=1}^n \int_\Omega \partial_j\left((\partial_j u)\overline{v}\right)dx + \sum_{j=1}^n \int_\Omega (\partial_j u)\left(\overline{\partial_j v}\right)dx \\
&= -\int_{\partial\Omega} (\overline{v}\nabla u, n_x)dx + (\nabla u, \nabla v)_{L^2} = (\nabla u, \nabla v)_{L^2},
\end{aligned}$$

where $\partial\Omega$ is the boundary of Ω and n_x is the unit outward vector at $x \in \partial\Omega$. Here we have made use of the divergence theorem. In a similar fashion we obtain

$$(\nabla u, \nabla v)_{L^2} = -\sum_{j=1}^{n} \int_{\Omega} u \partial_j^2 \overline{v} dx = (u, -\Delta v)_{L^2} = (u, Av)_{L^2}.$$

Hence $A \subset A^*$ and A is closable.

Definition. Let $s \geq 0$. The L^2-based *Sobolev space* of order s is defined by

$$H^s(\mathbb{R}^n) \equiv W_2^s(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi < \infty \right\},$$

where $\widehat{f}(\xi)$ is the Fourier transform of $f(x)$ given by

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(x,\xi)} f(x) dx.$$

The space $H^s(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_s := \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

By the symbol $\overset{\circ}{H}{}^s(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ with respect to the same norm.

If $s \in \mathbb{N}$, say $s = k$, then the norm $\|\cdot\|_s$ is equivalent to the norm

$$\|f\|_{W_2^k(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} |D^\alpha f|^2 dx,$$

where $D^\alpha f$ are the generalized derivatives in $L^2(\mathbb{R}^n)$. We also see the space $\overset{\circ}{W}{}_2^k(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W_2^k(\mathbb{R}^n)}^2$. Moreover, we can also consider $W_2^k(\Omega)$.

Example 6.2. Recall from Example 6.1 that

$$(-\Delta u, v)_{L^2} = (\nabla u, \nabla v)_{L^2}, \quad u, v \in C_0^\infty(\Omega).$$

Hence

$$(-\Delta u, u)_{L^2} = \|\nabla u\|_{L^2}^2 \leq \|u\|_{L^2} \|\Delta u\|_{L^2}, \quad u \in C_0^\infty(\Omega).$$

Therefore,

$$\begin{aligned} \|u\|_{W_2^2}^2 &= \|u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \\ &\leq \|u\|_{L^2}^2 + 2\|u\|_{L^2} \|\Delta u\|_{L^2} + \|\Delta u\|_{L^2}^2 \\ &\leq 2\|u\|_{L^2}^2 + 2\|\Delta u\|_{L^2}^2 \equiv 2\|u\|_A^2, \end{aligned}$$

where $\|\cdot\|_A$ is a norm which corresponds to the operator $A = -\Delta$ as follows:

$$\|u\|_A^2 := \|u\|_{L^2}^2 + \|-\Delta u\|_{L^2}^2 .$$

It is also clear that $\|u\|_A \le \|u\|_{W_2^2}$. Combining these inequalities gives

$$\frac{1}{\sqrt{2}} \|u\|_{W_2^2} \le \|u\|_A \le \|u\|_{W_2^2}$$

for all $u \in C_0^\infty(\Omega)$. A completion of $C_0^\infty(\Omega)$ with respect to these norms leads us to the statement:

$$D(\overline{A}) = \overset{\circ}{W}{}_2^2(\Omega).$$

Thus $\overline{A} = -\Delta$ on $D(\overline{A}) = \overset{\circ}{W}{}_2^2(\Omega)$. Let us determine $D(A^*)$ in this case. By the definition of $D(A^*)$ we have

$$D((-\Delta)^*) = \big\{v \in L^2(\Omega) : \text{there exists}\, v^* \in L^2(\Omega)\, \text{such that} \\ (-\Delta u, v) = (u, v^*)\, \text{for all}\, u \in C_0^\infty(\Omega)\} .$$

If we assume that $v \in W_2^2(\Omega)$ then it is equivalent to

$$(u, (-\Delta)^* v) = (u, v^*)$$

i.e. $(-\Delta)^* v = v^*$ and $D((-\Delta)^*) = W_2^2(\Omega)$. Finally, for $\Omega \subset \mathbb{R}^n$ with $\Omega \ne \mathbb{R}^n$ we obtain that

$$A \subset \overline{A} \subset A^* \equiv (\overline{A})^*$$

and $A \ne \overline{A}$ and $\overline{A} \ne (\overline{A})^*$, that is, the closure of A does not lead us to a self-adjoint operator.

Remark. If $\Omega = \mathbb{R}^n$ then $\overset{\circ}{W}{}_2^2(\mathbb{R}^n) \equiv W_2^2(\mathbb{R}^n)$ and therefore

$$\overline{A} = A^* = (\overline{A})^*.$$

Hence the closure of A is self-adjoint in that case.

Example 6.3. Consider again $A = -\Delta$ on $D(A) = C_0^\infty(\Omega)$ with $\Omega \ne \mathbb{R}^n$. Since

$$(-\Delta u, u)_{L^2} = \|\nabla u\|_{L^2}^2 \ge 0$$

then $-\Delta$ is non-negative with lower bound $\lambda = 0$. That's why

$$Q(u, v) := (\nabla u, \nabla v)_{L^2}$$

is a densely defined and non-negative quadratic form with $D(Q) \equiv D(A) = C_0^\infty(\Omega)$. A new inner product is defined as

$$(u, v)_Q := (\nabla u, \nabla v)_{L^2} + (u, v)_{L^2}$$

and

$$\|u\|_Q^2 \equiv \|u\|_{W_2^1(\Omega)}^2 .$$

If we apply now the procedure from Theorem 2 from Section 5 then we obtain the existence of $Q_1 = \overline{Q}$ with respect to the norm $\|\cdot\|_Q$ which will also be non-negative and closed with $D(Q_1) \equiv \overset{\circ}{W}{}_2^1(\Omega)$. Next step is to get the Friedrichs extension A_F as

$$A_F = J^{-1} - I$$

with $D(A_F) \equiv R(J) \subset \overset{\circ}{W}{}_2^1(\Omega)$. More careful examination of Theorem 1 of Section 5 leads us to the fact

$$D(A_F) = \overset{\circ}{W}{}_2^1(\Omega) \cap D(A^*) = \overset{\circ}{W}{}_2^1(\Omega) \cap W_2^2(\Omega).$$

Remark. In general, for symmetric operator, we have

$$D(A_F) = \{u \in H_Q : Au \in H\}$$

which is equivalent to

$$D(A_F) = \{u \in H_Q : u \in D(A^*)\} .$$

Exercise 35. Let $H = L^2(\Omega)$ and $A(x, D) = -\Delta + q(x)$, where $q(x) = \overline{q(x)}$ and $q(x) \in L^\infty(\Omega)$. Define $\overline{A}$, A^* and A_F.

Exercise 36. Let $f \in C_0^\infty(\Omega)$. Prove that

$$\|f\|_s^2 \le \varepsilon \|f\|_{s_1}^2 + \frac{1}{4\varepsilon} \|f\|_{s_2}^2 ,$$

where $\varepsilon > 0$ and $s_1 > s > s_2$ with $s_1 + s_2 \ge 2s$.

Consider now bounded $\Omega \subset \mathbb{R}^n$ and an elliptic operator $A(x, D)$ in Ω of the form

$$A(x, D) = \sum_{|\alpha|=|\beta|\le m/2} D^\alpha (a_{\alpha\beta}(x) D^\beta),$$

where $a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$ are real. Assume that there exists $C_0 > 0$ such that

$$|a_{\alpha\beta}(x)| \le C_0, \quad |\alpha|, |\beta| < \frac{m}{2}$$

for all $x \in \Omega$. Assume also that $A(x, D)$ is elliptic, that is,

$$(-1)^{m/2} \sum_{|\alpha|=|\beta|=m/2} a_{\alpha\beta}(x) \xi^\alpha \xi^\beta \ge \nu \sum_{|\alpha|=m/2} |\xi^\alpha|^2, \quad \nu > 0.$$

Theorem 1 (*Gårding's inequality*)**.** *Suppose that $A(x,D)$ is as above. Then for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that*

$$(Af,f)_{L^2(\Omega)} \geq (\nu - \varepsilon)\,\|f\|^2_{W_2^{m/2}(\Omega)} - C_\varepsilon\,\|f\|^2_{L^2(\Omega)}$$

for any $f \in C_0^\infty(\Omega)$.

Proof. Let $f \in C_0^\infty(\Omega)$. Then integration by parts yields

$$\begin{aligned}
(Af,f)_{L^2(\Omega)} &= \sum_{|\alpha|=|\beta|\leq m/2} \int_\Omega D^\alpha(a_{\alpha\beta}(x)D^\beta f)\overline{f}dx \\
&= \sum_{|\alpha|=|\beta|=m/2} \int_\Omega (-1)^{|\alpha|} a_{\alpha\beta}(x) D^\alpha \overline{f} D^\beta f dx \\
&+ \sum_{|\alpha|=|\beta|<m/2} \int_\Omega (-1)^{|\alpha|} a_{\alpha\beta}(x) D^\alpha \overline{f} D^\beta f dx \\
&\geq \nu \sum_{|\alpha|=m/2} \int_\Omega |D^\alpha f|^2 dx - C_0 \sum_{|\alpha|=|\beta|<m/2} \int_\Omega |D^\alpha f||D^\beta f| dx \\
&\geq \nu \sum_{|\alpha|\leq m/2} \int_\Omega |D^\alpha f|^2 dx - (C_0+\nu) \sum_{|\alpha|<m/2} \int_\Omega |D^\alpha f|^2 dx \\
&= \nu\,\|f\|^2_{W_2^{m/2}(\Omega)} - (C_0+\nu)\,\|f\|^2_{W_2^{m/2-1}(\Omega)}.
\end{aligned}$$

Next we make use of the following

Lemma. *For any $\varepsilon > 0$ and $0 < \delta \leq m/2$ there is $C_\varepsilon(\delta) > 0$ such that*

$$(1+|\xi|^2)^{m/2-\delta} \leq \varepsilon(1+|\xi|^2)^{m/2} + C_\varepsilon(\delta)$$

for any $\xi \in \mathbb{R}^n$.

Proof. Let $\varepsilon > 0$ and $0 < \delta \leq m/2$. If $(1+|\xi|^2)^\delta \geq \frac{1}{\varepsilon}$ then

$$(1+|\xi|^2)^{-\delta} \leq \varepsilon.$$

Hence

$$(1+|\xi|^2)^{m/2-\delta} \leq \varepsilon(1+|\xi|^2)^{m/2}$$

i.e. the claim holds for any positive constant $C_\varepsilon(\delta)$. For $(1+|\xi|^2)^\delta < \frac{1}{\varepsilon}$ we can get

$$(1+|\xi|^2)^{m/2-\delta} < \left(\frac{1}{\varepsilon}\right)^{\frac{m/2-\delta}{\delta}} \equiv C_\varepsilon(\delta).$$

□

Applying this lemma with $\delta = 1$ to the norm of W_2^s-spaces we may conclude that

$$\|f\|^2_{W_2^{m/2-1}(\Omega)} \leq \varepsilon_1 \|f\|^2_{W_2^{m/2}(\Omega)} + C_{\varepsilon_1} \|f\|^2_{L^2(\Omega)}$$

for any $\varepsilon_1 > 0$. Hence

$$\begin{aligned}
(Af, f)_{L^2(\Omega)} &\geq \nu \|f\|^2_{W_2^{m/2}(\Omega)} - (C_0 + \nu) \|f\|^2_{W_2^{m/2-1}(\Omega)} \\
&\geq \nu \|f\|^2_{W_2^{m/2}(\Omega)} - (C_0 + \nu)\varepsilon_1 \|f\|^2_{W_2^{m/2}(\Omega)} - (C_0 + \nu) C_{\varepsilon_1} \|f\|^2_{L^2(\Omega)} \\
&= (\nu - \varepsilon) \|f\|^2_{W_2^{m/2}(\Omega)} - C_\varepsilon \|f\|^2_{L^2(\Omega)}.
\end{aligned}$$

This proves the theorem. □

Corollary 1. *There exists a self-adjoint Friedrichs extension A_F of A with domain $D(A_F) = \overset{\circ}{W}{}_2^{m/2}(\Omega) \cap W_2^m(\Omega)$.*

Proof. It follows from Gårding's inequality that

$$(Af, f)_{L^2(\Omega)} \quad \geq \quad -C_\varepsilon \|f\|^2_{L^2(\Omega)}, \qquad f \in D(A).$$

This means that $A_\mu := A + \mu I$ is positive for $\mu > C_\varepsilon$ and therefore Theorem 2 of Section 5 gives us the existence of

$$(A_\mu)_F \equiv (A_F)_\mu = A_F + \mu I$$

with domain

$$D(A_F) = D((A_\mu)_F) = \overset{\circ}{W}{}_2^{m/2}(\Omega) \cap D(A^*),$$

where $\overset{\circ}{W}{}_2^{m/2}(\Omega)$ is the domain of the corresponding closed quadratic form (see Theorem 2). If Ω is bounded with smooth boundary $\partial\Omega$ then it can be proved that

$$D(A^*) = W_2^m(\Omega).$$

□

Gårding's inequality has two more consequences. Firstly,

$$\|(A_F)_\mu f\|_{L^2} \geq C_0 \|f\|_{L^2}, \quad C_0 > 0$$

so that

$$(A_F)_\mu^{-1} : L^2(\Omega) \to L^2(\Omega).$$

Secondly,

$$\|(A_F)_\mu f\|_{W_2^{-m/2}(\Omega)} \geq C_0' \|f\|_{W_2^{m/2}(\Omega)}, \quad C_0' > 0$$

so that

$$(A_F)_\mu^{-1} : L^2(\Omega) \to \overset{\circ}{W}{}_2^{m/2}(\Omega).$$

Corollary 2. *The spectrum $\sigma(A_F) = \{\lambda_j\}_{j=1}^{\infty}$ is the sequence of eigenvalues of finite multiplicity with only one accumulation point at $+\infty$. In short, $\sigma(A_F) = \sigma_d(A_F)$. The corresponding orthonormal system $\{\psi_j\}_{j=1}^{\infty}$ of eigenfunctions forms an orthonormal basis and*

$$A_F f \overset{L^2}{=} \sum_{j=1}^{\infty} \lambda_j (f, \psi_j)\psi_j$$

for any $f \in D(A_F)$.

Proof. We begin with a lemma.

Lemma. *The embedding*

$$\overset{\circ}{W}{}_2^{m/2}(\Omega) \hookrightarrow L^2(\Omega)$$

is compact.

Proof. It is enough to show that for any $\{\varphi_k\}_{k=1}^{\infty} \subset \overset{\circ}{W}{}_2^{m/2}(\Omega)$ with $\|\varphi_k\|_{W_2^{m/2}} \le 1$ there exists $\{\varphi_{j_k}\}_{k=1}^{\infty}$ which is a Cauchy sequence in $L^2(\Omega)$. Since Ω is bounded we have

$$|\widehat{\varphi_k}(\xi)| \le \|\varphi_k\|_{L^2} |\Omega|^{1/2}$$

i.e. $\widehat{\varphi_k}(\xi)$ is uniformly bounded. That's why there exists $\widehat{\varphi_{j_k}}(\xi)$ which converges pointwise in $\mathbb{R}^n$. Next,

$$\begin{aligned}
\|\varphi_{j_k} - \varphi_{j_m}\|_{L^2}^2 &= \int_{\mathbb{R}^n} |\widehat{\varphi_{j_k}}(\xi) - \widehat{\varphi_{j_m}}(\xi)|^2 d\xi \\
&= \int_{|\xi|<r} |\widehat{\varphi_{j_k}}(\xi) - \widehat{\varphi_{j_m}}(\xi)|^2 d\xi + \int_{|\xi|>r} |\widehat{\varphi_{j_k}}(\xi) - \widehat{\varphi_{j_m}}(\xi)|^2 d\xi \\
&\le \int_{|\xi|<r} |\widehat{\varphi_{j_k}}(\xi) - \widehat{\varphi_{j_m}}(\xi)|^2 d\xi \\
&+ \frac{1}{(1+r^2)^{m/2}} \int_{\mathbb{R}^n} (1+|\xi|^2)^{m/2} |\widehat{\varphi_{j_k}}(\xi) - \widehat{\varphi_{j_m}}(\xi)|^2 d\xi \\
&= \int_{|\xi|<r} |\widehat{\varphi_{j_k}}(\xi) - \widehat{\varphi_{j_m}}(\xi)|^2 d\xi + (1+r^2)^{-m/2} \|\varphi_{j_k} - \varphi_{j_m}\|_{W_2^{m/2}}^2 \\
&:= I_1 + I_2.
\end{aligned}$$

The first term I_1 tends to 0 as $k, m \to \infty$ by the dominated convergence theorem of Lebesgue for any fixed $r > 0$. The second term converges to 0 as $r \to \infty$ because $\|\varphi_{j_k} - \varphi_{j_m}\|_{W_2^{m/2}} \le 2$ □

Lemma gives us that

$$(A_\mu)_F^{-1} : L^2(\Omega) \to L^2(\Omega)$$

is a compact operator. Applying Riesz-Schauder and Hilbert-Schmidt theorems we get

1) $\sigma((A_\mu)_F^{-1}) = \{0, \mu_1, \mu_2, \ldots\}$ with $\mu_j \ge \mu_{j+1} > 0$ and $\mu_j \to 0$ as $j \to \infty$.

2) μ_j is of finite multiplicity

3) $(A_\mu)_F^{-1}\,\psi_j = \mu_j\psi_j$, where $\{\psi_j\}_{j=1}^\infty$ is an orthonormal system

4) $\{\psi_j\}_{j=1}^\infty$ forms an orthonormal basis in $L^2(\Omega)$.

Since $A_F\psi_j = \lambda_j\psi_j$ with $\lambda_j = \frac{1}{\mu_j} - \mu$ then we may conclude that

$$\sigma(A_F) = \{\lambda_j\}_{j=1}^\infty, \quad \lambda_j \le \lambda_{j+1}, \lambda_j \to \infty.$$

Moreover, λ_j has finite multiplicity and ψ_j are the corresponding eigenfunctions. We have also the following representation

$$(A_\mu)_F^{-1}\,f = \sum_{j=1}^\infty \mu_j(f,\psi_j)\psi_j, \quad f \in L^2(\Omega).$$

Exercise 37. Prove that

$$A_F f = \sum_{j=1}^\infty \lambda_j(f,\psi_j)\psi_j$$

for any $f \in D(A_F)$.

Now we may conclude that the corollary is proved. □

7 Spectral function

Let us consider a bounded domain $\Omega \subset \mathbb{R}^n$ and an elliptic differential operator $A(x, D)$ in Ω of the form

$$A(x, D) = \sum_{|\alpha|=|\beta|\le m/2} D^\alpha (a_{\alpha\beta}(x) D^\beta),$$

where $a_{\alpha\beta} = a_{\beta\alpha}$ are real, $a_{\alpha\beta} \in C^\infty(\Omega)$ and bounded for all α and β. We assume that

$$(-1)^{m/2} \sum_{|\alpha|=|\beta|=m/2} a_{\alpha\beta}(x) \xi^\alpha \xi^\beta \ge \nu |\xi|^m, \quad \nu > 0.$$

As it was proved above there exists at least one self-adjoint extension of A with $D(A) = C_0^\infty(\Omega)$, namely, the Friedrichs extension A_F with

$$D(A_F) = \overset{\circ}{W}{}_2^{m/2}(\Omega) \cap W_2^m(\Omega).$$

Let us consider an arbitrary self-adjoint extension $\widehat{A}$ of A. Without loss of generality we assume that $\widehat{A} \ge 0$. That's why $\widehat{A}$ has the spectral representation

$$\widehat{A} = \int_0^\infty \lambda dE_\lambda$$

with domain

$$D\left(\widehat{A}\right) = \left\{ f \in L^2(\Omega) : \int_0^\infty \lambda^2 d(E_\lambda f, f) < \infty \right\}.$$

In general case we have no such formula for $D\left(\widehat{A}\right)$ as for the Friedrichs extension A_F. But we can say that

$$\overset{\circ}{W}{}_2^m(\Omega) \subset D\left(\widehat{A}\right).$$

Indeed, since $a_{\alpha\beta}(x) \in C^\infty(\Omega)$ and bounded then $A(x, D)$ can be rewritten in the usual form

$$A(x, D) = \sum_{|\gamma|\le m} \widetilde{a_\gamma}(x) D^\gamma$$

with bounded coefficients. Hence

$$\|Af\|_{L^2(\Omega)} \le c \sum_{|\gamma|\le m} \|D^\gamma f\|_{L^2(\Omega)} \equiv c \|f\|_{W_2^m(\Omega)}.$$

This proves the embedding.

Theorem 1 (Gårding). *If $\widehat{A} = \widehat{A}^*$ then E_λ is an integral operator in $L^2(\Omega)$ such that*

$$E_\lambda f(x) = \int_\Omega \overline{\theta(x, y, \lambda)} f(y) dy,$$

where $\theta(x, y, \lambda)$ is called the spectral function *and has the properties*

1) $\theta(x, y, \lambda) = \overline{\theta(y, x, \lambda)}$

2)

$$\theta(x,y,\lambda)=\int_\Omega \theta(x,z,\lambda)\theta(z,y,\lambda)dz$$

and

$$\theta(x,x,\lambda)=\int_\Omega |\theta(x,z,\lambda)|^2dz\geq 0$$

3)

$$\sup_{x\in\Omega_1}\|\theta(x,\cdot,\lambda)\|_{L^2(\Omega)}\leq c_1\lambda^k,$$

where $\overline{\Omega_1}=\Omega_1\subset\Omega, k\in\mathbb{N}$ *with* $k>\frac{n}{2m}$ *and* $c_1=c(\Omega_1)$.

Remark. It was proved by L. Hörmander that actually

$$\theta(x,x,\lambda)\leq c_1\lambda^{n/m}.$$

Corollary. *Let* $z\in\rho\left(\widehat{A}\right)$. *Then* $(\widehat{A}-zI)^{-1}$ *is an integral operator whose kernel* $G(x,y,z)$ *is called the* Green's function *corresponding to* $\widehat{A}$ *and which has the properties*

1)

$$G(x,y,z)=\int_0^\infty \frac{d_\lambda\overline{\theta(x,y,\lambda)}}{\lambda-z}$$

2) $\overline{G(x,y,z)}=G(y,x,\overline{z})$.

Proof. Since $z\in\rho\left(\widehat{A}\right)$ then J. von Neumann's spectral theorem gives us

$$(\widehat{A}-zI)^{-1}f=\int_0^\infty(\lambda-z)^{-1}dE_\lambda f.$$

Next, by Theorem 1 we get

$$\begin{aligned}(\widehat{A}-zI)^{-1}f &= \int_0^\infty(\lambda-z)^{-1}d_\lambda\left(\int_\Omega\overline{\theta(x,y,\lambda)}f(y)dy\right)\\ &= \int_\Omega\left(\int_0^\infty(\lambda-z)^{-1}d_\lambda\overline{\theta(x,y,\lambda)}\right)f(y)dy\\ &= \int_\Omega G(x,y,z)f(y)dy,\end{aligned}$$

where $G(x,y,z)$ is as in 1). Since

$$\overline{G(x,y,z)}=\int_0^\infty\frac{d\theta(x,y,\lambda)}{\lambda-\overline{z}}=\int_0^\infty\frac{d\overline{\theta(y,x,\lambda)}}{\lambda-\overline{z}}=G(y,x,\overline{z})$$

then 2) is also proved. $\square$

Exercise 38. Prove that $\theta(x, x, \lambda)$ is a monotone increasing function with respect to λ and

1) $|\theta(x, y, \lambda)|^2 \le \theta(x, x, \lambda)\theta(y, y, \lambda)$

2) $|E_\lambda f(x)| \le \theta(x, x, \lambda)^{1/2} \|f\|_{L^2(\Omega)}$.

Exercise 39. Prove that

$$|E_\lambda f(x) - E_\mu f(x)| \le \|E_\lambda f - E_\mu f\|_{L^2(\Omega)} |\theta(x, x, \lambda) - \theta(x, x, \mu)|^{1/2}$$

for any $\lambda > 0$ and $\mu > 0$.

Exercise 40. Let us assume that $n < m$. Prove that

$$G(x, y, z) = \int_0^\infty \frac{\overline{\theta(x, y, \lambda)} d\lambda}{(\lambda - z)^2}$$

and that $G(\cdot, y, z) \in L^2(\Omega)$.

8 Fundamental solution

Let us consider an elliptic differential operator of even order m in general form

$$A(x,D)=\sum_{|\alpha|\le m}a_\alpha(x)D^\alpha,$$

where $a_\alpha(x)\in C^\infty(\Omega)$ and bounded, $a_\alpha(x)$ real for $|\alpha|=m$ and

$$a(x,\xi)=\sum_{|\alpha|=m}a_\alpha(x)\xi^\alpha>0$$

for any $x\in\Omega$ and $\xi\in\mathbb{R}^n\setminus\{0\}$. Let us define the set

$$\mathbb{Z}_\theta:=\{z\in\mathbb{C}:|\arg z|<\theta\}$$

for some fixed $0<\theta<\pi/2$.

Lemma 1. *There is a constant $c_0>0$ such that, for any $z\notin\mathbb{Z}_\theta$,*

$$a(x,\xi)+1+|z|\ge|a(x,\xi)+1-z|\ge c_0\left(a(x,\xi)+1+|z|\right),$$

where $x\in\Omega$ and $\xi\in\mathbb{R}^n$.

Proof. Let $z=\lambda+i\mu\notin\mathbb{Z}_\theta$. Abbreviate $a:=a(x,\xi)\ge0$. The first (leftmost) inequality follows immediately from triangle inequality. Hence it remains to prove the latter (rightmost) inequality. To that end, we start from

$$|a+1-z|^2=(a+1)^2-2\lambda(a+1)+\lambda^2+\mu^2.$$

If $\lambda\le0$ then

$$\begin{aligned}(a+1)^2-2\lambda(a+1)+\lambda^2+\mu^2&=(a+1)^2+2|\lambda|(a+1)+\lambda^2+\mu^2\\&\ge(a+1)^2+\lambda^2+\mu^2=(a+1)^2+|z|^2\\&\ge\frac{1}{2}\left(a+1+|z|\right)^2.\end{aligned}$$

Consider now $\lambda>0$. Since $z\notin\mathbb{Z}_\theta$ then $|\mu|\ge|h|=\lambda\tan\theta$, see Figure 1. It means that $\mu^2\ge\gamma\lambda^2$, where $\gamma=\tan^2\theta>0$. Hence

$$\begin{aligned}(a+1)^2-2\lambda(a+1)+\lambda^2+\mu^2&\ge(a+1)^2-\varepsilon(a+1)^2-\frac{1}{\varepsilon}\lambda^2+\lambda^2+\mu^2\\&=(1-\varepsilon)(a+1)^2+\left(1-\frac{1}{\varepsilon}\right)\lambda^2+\delta\mu^2+(1-\delta)\mu^2\\&\ge(1-\varepsilon)(a+1)^2+\left(1-\frac{1}{\varepsilon}\right)\lambda^2+\gamma\delta\lambda^2+(1-\delta)\mu^2\\&=(1-\varepsilon)(a+1)^2+\left(1-\frac{1}{\varepsilon}+\gamma\delta\right)\lambda^2+(1-\delta)\mu^2\end{aligned}$$

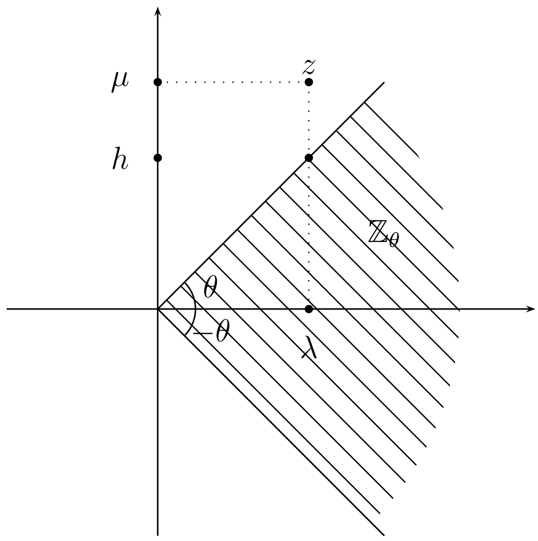

Figure 1: The geometry of Lemma 1.

for any $\varepsilon > 0$ and $\delta \in \mathbb{R}$. Let us choose $\delta = \frac{1}{2}$ and ε such that

$$\frac{2}{2+\gamma} < \varepsilon < 1.$$

Then for

$$2c_0^2 := \min\left(\frac{1}{2}, 1-\varepsilon, 1-\frac{1}{\varepsilon}+\frac{\gamma}{2}\right) > 0$$

we have that

$$|a+1-z|^2 \geq 2c_0^2\left((a+1)^2+\lambda^2+\mu^2\right) = 2c_0^2\left((a+1)^2+|z|^2\right) \geq c_0^2\left(a+1+|z|\right)^2.$$

This proves the lemma. □

Let $\varphi_k(x,\xi,z), k = 0,1,\ldots$ be the sequence defined by

$$\begin{aligned}
\varphi_0(x,\xi,z) &= \frac{1}{a(x,\xi)+1-z}, \\
\varphi_k(x,\xi,z) &= \frac{1}{a(x,\xi)+1-z}\left[a(x,\xi)+1-A(x,D+\xi)\right]\varphi_{k-1}, \quad k = 1,2,\ldots,
\end{aligned}$$

where

$$A(x,D+\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)(D+\xi)^\alpha$$

with $x \in \Omega, \xi \in \mathbb{R}^n$ and $z \notin \mathbb{Z}_\theta$. Due to Lemma 1 this sequence is well-defined.

Exercise 41. Prove that

$$(D+\xi)^\alpha f = \sum_{\beta\leq\alpha} C_\alpha^\beta \xi^{\alpha-\beta} D^\beta f.$$

Exercise 42. Prove that

$$A(x,D)(fg) = \sum_{\alpha \geq 0} \frac{D^\alpha f}{\alpha!} A^{(\alpha)}(x,D)g,$$

where $A^{(\alpha)}$ has the symbol $A^{(\alpha)}(x,\xi) = \partial_\xi^\alpha A(x,\xi)$.

Hint. Check these first for $f = e^{i(x,\eta)}$ and $g = e^{i(x,\xi)}$.

Lemma 2. *Every $\varphi_k, k = 0, 1, \ldots$ is a finite sum of terms of the form*

$$\frac{b(x,\xi)}{(a+1-z)^{\nu+1}}, \quad \nu = 0, 1, \ldots,$$

where $b(x,\xi)$ is a homogeneous polynomial of order l with respect to ξ and $m\nu - l \geq k$.

Proof. We prove by induction with respect to k.

For $k = 0$ the claim is immediate because

$$\varphi_0(x,\xi,z) = \frac{1}{a(x,\xi)+1-z}$$

allows us to take $\nu = 0, l = 0$ so that $m \cdot 0 - 0 \geq 0$.

Assume that the claim holds for $0 \leq k \leq j$ i.e.

$$\varphi_k = \sum_\nu \frac{b(x,\xi)}{(a+1-z)^{\nu+1}}$$

and $m\nu - l \geq k$. Concerning φ_{j+1} we have that

$$\begin{aligned}
\varphi_{j+1} &= \frac{1}{a+1-z}\left[a+1-A(x,D+\xi)\right]\varphi_j \\
&= \frac{(a+1)\varphi_j}{a+1-z} - \frac{1}{a+1-z}\sum_{|\alpha|\leq m} a_\alpha(x) \sum_{\beta \leq \alpha} C_\alpha^\beta \xi^{\alpha-\beta} D^\beta \varphi_j \\
&= \frac{(a+1)\varphi_j}{a+1-z} - \frac{1}{a+1-z}\sum_{|\beta|\leq m}\sum_{|\beta+\gamma|\leq m} a_{\beta+\gamma}(x) C_{\beta+\gamma}^\beta \xi^\gamma D^\beta \varphi_j \\
&:= I_1 + I_2,
\end{aligned}$$

where $I_1 = \frac{(a+1)\varphi_j}{a+1-z}$. The term in I_2 that corresponds to $\beta = 0$ and $|\gamma| = m$ is

$$-\frac{1}{a+1-z}\sum_{|\gamma|=m} a_\gamma(x)\xi^\gamma \varphi_j = -\frac{a\varphi_j}{a+1-z}.$$

Hence $I_1 + I_2$ can be rewritten as

$$\begin{aligned}
I_1 + I_2 &= \frac{(a+1)\varphi_j}{a+1-z} - \frac{a\varphi_j}{a+1-z} - \frac{1}{a+1-z}\sum_{\substack{|\beta|\leq m \\ \beta > 0}}\sum_{|\beta+\gamma|\leq m} a_{\beta+\gamma}(x) C_{\beta+\gamma}^\beta \xi^\gamma D^\beta \varphi_j \\
&= I_1' + I_2',
\end{aligned}$$

where

$$I_1' = \frac{\varphi_j}{a+1-z}.$$

Since

$$\varphi_j = \sum_\nu \frac{b(x,\xi)}{(a+1-z)^{\nu+1}}$$

then

$$I_1' = \sum_\nu \frac{b(x,\xi)}{(a+1-z)^{\nu+2}}$$

and $m\nu - l \geq j$. This implies that $m\nu + m - l \geq j + m > j + 1$, since $m \geq 2$. In other words,

$$m(\nu+1) - l \geq j+1$$

and the lemma is proved for I_1'. Let us consider now $D^\beta \varphi_j$ from I_2' i.e.

$$D^\beta \varphi_j = D^\beta \sum_\nu \frac{b(x,\xi)}{(a+1-z)^{\nu+1}} = \sum_\nu D_x^\beta \frac{b(x,\xi)}{(a+1-z)^{\nu+1}}, \quad m\nu - l \geq j.$$

We prove that differentiation of

$$\frac{b(x,\xi)}{(a+1-z)^{\nu+1}}$$

with respect to x leads to

$$\sum_{\widetilde{\nu}} \frac{\widetilde{b}}{(a+1-z)^{\widetilde{\nu}+1}},$$

where $m\widetilde{\nu} - \widetilde{l} = m\nu - l \geq j$ i.e. the value of $m\nu - l$ does not change. Indeed,

$$\partial_j \frac{b}{(a+1-z)^{\nu+1}} = \frac{b_j'}{(a+1-z)^{\nu+1}} - \frac{(\nu+1)ba_j'}{(a+1-z)^{\nu+2}},$$

where b_j' and ba_j' are homogeneous polynomials with respect to ξ of orders l and $\widetilde{l} = l + m$, respectively. That's why

$$m\widetilde{\nu} - \widetilde{l} = m(\nu+1) - (m+l) = m\nu - l.$$

This fact holds for any derivative $D^\beta, \beta > 0$. Now we may conclude that

$$I_2' = -\frac{1}{a+1-z} \sum_{\widetilde{\nu}} \sum_{\substack{|\beta| \leq m \\ \beta > 0}} \sum_{|\beta+\gamma| \leq m} C_{\beta+\gamma}(x) \xi^\gamma \frac{\widetilde{b}}{(a+1-z)^{\widetilde{\nu}+1}},$$

where $|\gamma| \leq m-1$ and $m\widetilde{\nu} - \widetilde{l} = m\nu - l \geq j$. But multiplication by $\xi^\gamma, |\gamma| \leq m-1$ leads us finally to the sum of the terms

$$\frac{1}{(a+1-z)} \frac{\widetilde{b}}{(a+1-z)^{\widetilde{\nu}+1}} \xi^\gamma = \frac{\widetilde{b}\xi^\gamma}{(a+1-z)^{\widetilde{\nu}+2}} = \frac{\widetilde{b_1}}{(a+1-z)^{\widetilde{\nu}+2}},$$

where $\widetilde{l_1} = \widetilde{l} + |\gamma|$ and

$$m(\widetilde{\nu}+1) - \widetilde{l_1} = m\widetilde{\nu} + m - \widetilde{l} - |\gamma| = m\nu - l + m - |\gamma| \geq m\nu - l + 1 \geq j+1.$$

This finishes the proof. $\square$

Definition. A locally integrable function $F(\cdot, y, z)$ with parameters $y \in \Omega$ and $z \notin \mathbb{Z}_\theta$ is called a *fundamental solution* of the operator $A(\cdot, D) - zI$ in Ω if

$$(A(\cdot, D) - zI)F(\cdot, y, z) = \delta(\cdot - y)$$

in the sense of distributions i.e.

$$\langle (A - zI)F(\cdot, y, z), \varphi(\cdot)\rangle = \varphi(y)$$

for all $\varphi \in C_0^\infty(\Omega)$ or

$$\int_\Omega F(x, y, z)(A^* - zI)\varphi(x)dx = \varphi(y),$$

where $A^*(x, D)$ is formally adjoint to $A(x, D)$ in $L^2(\Omega)$.

Let us define a new function

$$F_k(x, y, z) := F^{-1}\left(\sum_{j=0}^{k} \varphi_j(x, \cdot, z)\right)(x - y), \quad k = 0, 1, 2, \ldots.$$

Here F^{-1} is the inverse Fourier transform of tempered distributions i.e.

$$\langle F^{-1}u, \varphi\rangle := \langle u, F^{-1}\varphi\rangle,$$

where $\varphi \in C^\infty(\mathbb{R}^n)$ is such that all of its derivatives decay faster than the reciprocal of any polynomial at infinity and

$$(F^{-1}f)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(x,y)} f(y)dy$$

for $f \in L^1(\mathbb{R}^n)$. Then the following lemma holds.

Lemma 3. *In the sense of distributions, the function $F_k(x, y, z)$ satisfies*

$$(A(x, D) - zI)F_k(x, y, z) = \delta(x - y) - H_k(x, y, z),$$

where $H_k(x, y, z) = F^{-1}((a(x, \cdot) + 1 - z)\varphi_{k+1})$.

Proof. Since

$$\begin{aligned} D_x^\alpha(\varphi_j(x, \xi, z)e^{i(x-y,\xi)}) &= \sum_{\beta\leq\alpha} C_\alpha^\beta \left(D_x^\beta \varphi_j\right)\left(D_x^{\alpha-\beta}e^{i(x-y,\xi)}\right) \\ &= \left(\sum_{\beta\leq\alpha} C_\alpha^\beta \xi^{\alpha-\beta} D_x^\beta \varphi_j(x, \xi, z)\right) e^{i(x-y,\xi)} \end{aligned}$$

then

$$\begin{aligned}
(A(x,D)-zI)F_k(x,y,z) &= \sum_{j=0}^{k} F^{-1}\left((A(x,D+\cdot)-zI)\varphi_j(x,\cdot,z)\right)\\
&= \sum_{j=0}^{k} F^{-1}\left((a+1-z)\varphi_j-(a+1-A(x,D+\cdot))\varphi_j\right)\\
&= \sum_{j=0}^{k} F^{-1}\left((a+1-z)\varphi_j-(a+1-z)\varphi_{j+1}\right)\\
&= F^{-1}\left((a+1-z)\varphi_0-(a+1-z)\varphi_{k+1}\right)\\
&= F^{-1}(1)-F^{-1}\left((a+1-z)\varphi_{k+1}\right)\\
&= \delta(x-y)-H_k(x,y,z).
\end{aligned}$$

This proves the lemma. □

Remark. The following representation holds:

$$H_k(x,y,z)=\sum_{\nu} F^{-1}\left(\frac{b_\nu(x,\cdot)}{(a(x,\cdot)+1-z)^{\nu}}\right),$$

where $\nu = 1, 2, \ldots$ and $m\nu - l_\nu \geq k+1$.

Next, we look for the fundamental solution $F(x,y,z)$ of $A(x,D)-zI$ in the form

$$F(x,y,z)=F_k(x,y,z)+\int_{\Omega_1} F_k(x,u,z)h_k(u,y,z)du,$$

where $x, y \in \overline{\Omega_1} = \Omega_1 \subset \Omega$, $z \notin \mathbb{Z}_\theta$ and $h_k(u,y,z)$ is an unknown function which is to be determined. Since

$$(A-zI)F_k=\delta-H_k$$

then

$$(A-zI)F=\delta-H_k+\int_{\Omega_1}(\delta-H_k)h_k du=\delta-H_k+h_k-\int_{\Omega_1}H_k h_k du.$$

Hence F is the fundamental solution if and only if h_k satisfies the equation

$$h_k(x,y,z)=H_k(x,y,z)+\int_{\Omega_1}H_k(x,u,z)h_k(u,y,z)du.$$

We plan to solve this equation by iterations. In order to do it and obtain estimates of $F(x,y,z)$ it will be necessary to estimate

$$F_k=\sum_{j=0}^{k}F^{-1}\varphi_j=\sum_j\sum_{\nu_j}F^{-1}\left(\frac{b_{\nu_j}}{(a+1-z)^{\nu_j+1}}\right),$$

where $\nu_j = 0, 1, \dots$ and $m\nu_j - l_j \geq j$ and

$$H_k = \sum_{\nu} F^{-1}\left(\frac{b_\nu}{(a+1-z)^\nu}\right),$$

where $\nu = 1, 2, \dots$ and $m\nu - l \geq k+1$. Let us introduce the function

$$\psi(x, u) := F^{-1}\left(\frac{b(x,\cdot)}{(a(x,\cdot)+1)^\nu}\right)(u),$$

where $m\nu - l > 0$. This definition is understood in the sense of tempered distributions i.e.

$$\langle \psi(x,\cdot), \varphi\rangle = \left\langle \frac{b}{(a+1)^\nu}, F^{-1}\varphi \right\rangle = \int_{\mathbb{R}^n} \frac{b(x,\xi)}{(a(x,\xi)+1)^\nu} F^{-1}\varphi(\xi) d\xi.$$

Lemma 4 (Main lemma)**.** *Suppose that $|u| = 1$. For every $\overline{\Omega_1} = \Omega_1 \subset \Omega$ there is $\delta > 0$ such that for any $\lambda > 0$ we have*

1) $m\nu - l < n$,

$$|\psi(x, \lambda u)| \leq c_1 \lambda^{m\nu - l - n} e^{-\delta\lambda};$$

2) $m\nu - l = n$,

$$|\psi(x, \lambda u)| \leq c_1 (1 + |\log \lambda|) e^{-\delta\lambda};$$

3) $m\nu - l > n$,

$$|\psi(x, \lambda u)| \leq c_1 e^{-\delta\lambda},$$

where $x \in \Omega_1$ and $c_1 = c(\Omega_1)$.

Proof. Let us denote by c_1 any constant that depends on Ω_1.

Since $a(x,\xi) > 0$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ then for any $\overline{\Omega_1} = \Omega_1 \subset \Omega$ there is a constant $c_1 > 0$ such that

$$a(x,\xi) \geq c_1 |\xi|^m, \quad x \in \Omega_1, \xi \in \mathbb{R}^n.$$

This way we can consider the function

$$\psi_0(x, u) := F^{-1}\left(\frac{b(x,\cdot)}{(a(x,\cdot))^\nu}\right)(u),$$

where $x \in \Omega_1$ and $u \in \mathbb{R}^n$. It is very easy to check that

$$\psi_0(x, \lambda u) = \lambda^{m\nu - l - n} \psi_0(x, u).$$

1) Let $m\nu - l < n$. Let us prove first that

$$|\psi_0(x,u)| \le c_1$$

for $|u| = 1$ and $x \in \Omega_1$. Take a function $\chi(\xi) \in C_0^\infty(\mathbb{R})$ such that $\chi(\xi) \equiv 1$ for $|\xi| \le 1$. Then

$$\psi_0(x,u) = F^{-1}\left(\frac{b}{a^\nu}\chi\right) + F^{-1}\left(\frac{b}{a^\nu}(1-\chi)\right) := \widetilde{\psi_0} + \widetilde{\widetilde{\psi_0}}.$$

Since $m\nu - l < n$ then $\frac{b}{a^\nu} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. That's why

$$\widetilde{\psi_0}(x,u) = F^{-1}\left(\frac{b}{a^\nu}\chi\right) = (2\pi)^{-n}\int_{|\xi|<L}\frac{b(x,\xi)}{a(x,\xi)^\nu}\chi(\xi)e^{i(\xi,u)}d\xi,$$

where $\operatorname{supp}\chi \subset \{|\xi| < L\}$. Hence

$$|\widetilde{\psi_0}(x,u)| \le c_1\int_{|\xi|<L}|\xi|^{l-m\nu}d\xi = c_1'.$$

For $\widetilde{\widetilde{\psi_0}}(x,u)$ we have the identity

$$u^\alpha\widetilde{\widetilde{\psi_0}}(x,u) = F^{-1}\left(D_\xi^\alpha\left[\frac{b}{a^\nu}(1-\chi)\right]\right).$$

Next, we prove that

$$\left|D_\xi^\alpha\left[\frac{b}{a^\nu}(1-\chi(\xi))\right]\right| \le \frac{c_1}{(1+|\xi|)^{m\nu-l+|\alpha|}}.$$

Indeed,

$$D_{\xi_j}\left[\frac{b}{a^\nu}(1-\chi(\xi))\right] = \frac{b'_{\xi_j}}{a^\nu}(1-\chi) - \frac{\nu b a'_{\xi_j}}{a^{\nu+1}}(1-\chi) - \frac{b}{a^\nu}\chi' := I_1 + I_2 + I_3.$$

It is clear that $I_3 \equiv 0$ for $|\xi| < 1$ and $|\xi| > L$. Hence $|I_3| \le c_1$ and has compact support. For $|\xi| > 1$ the term I_1 satisfies

$$|I_1| \le c_1\frac{|\xi|^{l-1}}{|\xi|^{m\nu}} = c_1|\xi|^{l-m\nu-1}.$$

The same estimate holds for I_2 too. Since $I_1 \equiv 0$ and $I_2 \equiv 0$ for $|\xi| < 1$ then combining these estimates we obtain

$$\left|D_{\xi_j}\left[\frac{b}{a^\nu}(1-\chi(\xi))\right]\right| \le \frac{c_1}{(1+|\xi|)^{m\nu-l+1}}.$$

The required estimate follows now by induction. If we choose now α such that $|\alpha| > n + l - m\nu$ i.e. $m\nu - l + |\alpha| > n$ then

$$\left|u^\alpha\widetilde{\widetilde{\psi_0}}(x,u)\right| \le c_1\int_{\mathbb{R}^n}\frac{d\xi}{(1+|\xi|)^{m\nu-l+|\alpha|}} = c_1'.$$

Since we can choose coordinates so that $u = (|u|, 0, \dots, 0)$ we obtain

$$\left|\widetilde{\widetilde{\psi_0}}(x,u)\right| \le c_1, \quad x \in \Omega_1, |u| = 1.$$

The same arguments as above lead us also to

$$|\psi(x,u)| \le c_1, \quad x \in \Omega_1, |u| = 1.$$

Let us consider now the main symbol $a(x,\zeta)$ for complex ζ. Let $\zeta = \xi + i\eta$. Then

$$\begin{aligned} a(x,\zeta) &= \sum_{|\alpha|=m} a_\alpha(x)(\xi + i\eta)^\alpha = \sum_{|\alpha|=m} a_\alpha(x) \sum_{\beta\le\alpha} C_\alpha^\beta \xi^\beta (i\eta)^{\alpha-\beta} \\ &= \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha + \sum_{|\alpha|=m} a_\alpha(x) \sum_{\beta<\alpha} C_\alpha^\beta \xi^\beta (i\eta)^{\alpha-\beta}. \end{aligned}$$

For $|\eta| \le \delta$ it follows that

$$\begin{aligned} |a(x,\zeta)+1| &\ge a(x,\xi) + 1 - \sum_{|\alpha|=m} a_\alpha(x) \sum_{\beta<\alpha} C_\alpha^\beta |\xi|^{|\beta|} |\eta|^{|\alpha-\beta|} \\ &\ge a(x,\xi) + 1 - c_1 \sum_{j=0}^{m-1} |\xi|^j |\eta|^{m-j} \\ &\ge a(x,\xi) + 1 - c_1 \sum_{j=0}^{m-1} \left[\varepsilon|\xi|^m + \frac{1}{\varepsilon^{j/(m-j)}}|\eta|^m\right] \\ &= a(x,\xi) + 1 - c_1\varepsilon m|\xi|^m - c_\varepsilon|\eta|^m \\ &\ge \widetilde{c_1}|\xi|^m - c_1\varepsilon m|\xi|^m + 1 - c_\varepsilon|\eta|^m \\ &= (\widetilde{c_1} - \varepsilon c_1 m)|\xi|^m + 1 - c_\varepsilon|\eta|^m \ge c_0(|\xi|^m + 1), \end{aligned}$$

where

$$c_0 = \min\{\widetilde{c_1} - \varepsilon c_1 m, 1 - c_\varepsilon \delta^m\} > 0$$

and the parameters are chosen as

$$\widetilde{c_1} - \varepsilon c_1 m > 0, \quad 1 - c_\varepsilon\delta^m > 0$$

i.e.

$$\varepsilon < \frac{\widetilde{c_1}}{c_1 m}, \quad \delta < \left(\frac{1}{c_\varepsilon}\right)^{\frac{1}{m}}.$$

We proved that there exists $\delta > 0$ such that

$$|a(x,\xi+i\eta)+1| \ \ge \ c_0(|\xi|^m + 1), \quad c_0 > 0$$

for all $\xi, \eta \in \mathbb{R}^n, |\eta| \le \delta$ and $x \in \Omega_1$. This inequality allows us to extend the function

$$D_\xi^\alpha \left[\frac{b(x,\xi)}{(a(x,\xi)+1)^\nu}\right] e^{i(\lambda u,\xi)}$$

as an analytic function with respect to $\zeta = \xi + i\eta$ for $\xi \in \mathbb{R}^n$ and $|\eta| \le \delta$. Moreover, this inequality leads to the estimate

$$\left| D_\xi^\alpha \left[\frac{b(x,\xi)}{(a(x,\xi)+1)^\nu} \right] \right| \le \frac{c_1}{(1+|\xi|)^{m\nu - l + |\alpha|}}$$

for all $x \in \Omega_1, \xi \in \mathbb{R}^n$ and $|\eta| \le \delta$. Let us consider fixed $\eta \in \mathbb{R}^n$ with $|\eta| \le \delta$. Cauchy theorem for $|\alpha| > l - m\nu + n$ shows us that

$$\begin{aligned}(\lambda u)^\alpha \psi(x, \lambda u) &= (2\pi)^{-n} \int_{\mathbb{R}^n} D_\xi^\alpha \frac{b(x,\xi)}{(a(x,\xi)+1)^\nu} e^{i(\lambda u, \xi)} d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} D_\xi^\alpha \frac{b(x,\xi + i\eta)}{(a(x,\xi+i\eta)+1)^\nu} e^{i(\lambda u, \xi + i\eta)} d\xi.\end{aligned}$$

Hence

$$|(\lambda u)^\alpha \psi(x, \lambda u)| \le c_1 \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^{m\nu - l + |\alpha|}} e^{-\lambda(u,\eta)} d\xi \le c_1' e^{-\lambda(u,\eta)}.$$

Since $|u^\alpha| = |u_1|^{\alpha_1} \cdots |u_n|^{\alpha_n} > 0$ for $|u| = 1$ then $|u^\alpha| \ge \min_{|u|=1} |u^\alpha| = |u_0^\alpha| := c_0$, where $|u_0| = 1$ and $c_0 > 0$. Therefore the latter inequality can be transformed to

$$|\psi(x, \lambda u)| \le c_1'' \lambda^{-|\alpha|} e^{-\lambda(u,\eta)}.$$

If we choose $\eta = \delta u$ then we obtain

$$|\psi(x, \lambda u)| \le c_1 \lambda^{-|\alpha|} e^{-\delta\lambda}, \quad \lambda > 0, |u| = 1, x \in \Omega_1,$$

where $|\alpha| > l - m\nu + n$. For $|\alpha| = l - m\nu + n + 1 > 0$ and for $\lambda \ge 1$ we can obtain

$$|\psi(x, \lambda u)| \le c_1 \lambda^{m\nu - l - n - 1} e^{-\delta\lambda} \le c_1 \lambda^{m\nu - l - n} e^{-\delta\lambda}.$$

Consider now $0 < \lambda < 1$. It is easily seen that

$$\begin{aligned}(\lambda u)^\alpha (\psi(x, \lambda u) - \psi_0(x, \lambda u)) &= F^{-1} \left(D_\xi^\alpha \left[\frac{b_\nu}{(a+1)^\nu} - \frac{b_\nu}{a^\nu} \right] \right) (\lambda u) \\ &= -F^{-1} \left(D_\xi^\alpha \left[\frac{(a+1)^\nu - a^\nu}{(a^2+a)^\nu} b_\nu \right] \right) (\lambda u) \\ &= -F^{-1} \left(D_\xi^\alpha \left[\frac{\sum_{j=0}^{\nu-1} c_\nu^j a^j b_\nu}{(a^2+a)^\nu} \right] \right) (\lambda u) \\ &= -F^{-1} \left(D_\xi^\alpha \left[\frac{\sum_{j=0}^{\nu-1} \widetilde{b_j}}{(a^2+a)^\nu} \right] \right) (\lambda u),\end{aligned}$$

where $\widetilde{b_j}$ is a homogeneous polynomial of order $\widetilde{l_j} = l + mj$. Let us denote

$$I(x,\xi) := \frac{\sum_{j=0}^{\nu-1} \widetilde{b_j}}{(a^2+a)^\nu}.$$

It is clear that $I(x,\xi)$ behaves as

$$I \asymp |\xi|^{l-m\nu}, \quad |\xi| \to 0$$

and

$$I \asymp |\xi|^{l-m\nu-m}, \quad |\xi| \to \infty.$$

Combining these two asymptotics yields

$$I \asymp \frac{1}{|\xi|^{m\nu-l}} \frac{1}{(1+|\xi|)^m}$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$. By the same arguments we have

$$|D^\alpha I| \asymp \frac{1}{|\xi|^{m\nu-l+|\alpha|}} \frac{1}{(1+|\xi|)^m}.$$

Let us choose now $|\alpha| = l - m\nu + n - 1 \geq 0$. Then

$$|D^\alpha I| \asymp \frac{1}{|\xi|^{n-1}} \frac{1}{(1+|\xi|)^m}.$$

This allows us to get $D^\alpha I \in L^1(\mathbb{R}^n)$ and, therefore,

$$|(\lambda u)^\alpha(\psi(x,\lambda u) - \psi_0(x,\lambda u))| \leq c_1 \int_{\mathbb{R}^n} \frac{1}{|\xi|^{n-1}(1+|\xi|)^m} \leq c_1'.$$

Hence

$$|\psi(x,\lambda u) - \psi_0(x,\lambda u)| \leq c_1 \lambda^{m\nu-l-n+1}$$

for $|u| = 1$. It implies that

$$\begin{aligned} |\psi(x,\lambda u)| &\leq c_1\lambda^{m\nu-l-n+1} + |\psi_0(x,\lambda u)| \leq c_1\lambda^{m\nu-l-n} + \lambda^{m\nu-l-n}|\psi_0(x,u)| \\ &\leq c_1\lambda^{m\nu-l-n} \leq c_1'\lambda^{m\nu-l-n}e^{-\delta\lambda}, \quad 0<\lambda<1. \end{aligned}$$

Thus, 1) is proved for all $\lambda > 0$.

2) Let $m\nu - l = n$. Taking the derivative with respect to λ we obtain

$$\frac{\partial}{\partial\lambda}\psi(x,\lambda u) = F^{-1}\left(\frac{i(u,\xi)b(x,\xi)}{(a(x,\xi)+1)^\nu}\right)(\lambda u).$$

It is equivalent to

$$\frac{\partial}{\partial\lambda}\psi(x,\lambda u) = F^{-1}\left(\frac{\widetilde{b}(x,u,\xi)}{(a(x,\xi)+1)^\nu}\right)(\lambda u) := \widetilde{\psi}(x,\lambda u),$$

where $m\nu - \widetilde{l} = m\nu - l - 1 = n - 1 < n$. That's why we can apply part 1) to $\widetilde{\psi}(x,\lambda u)$ and obtain

$$\left|\frac{\partial}{\partial\lambda}\psi(x,\lambda u)\right| \leq c_1\lambda^{-1}e^{-\delta\lambda}.$$

Next, consider two cases: $0 < \lambda < 1$ and $\lambda \geq 1$. In the first case

$$\psi(x, \lambda u) = -\int_\lambda^1 \frac{\partial}{\partial \tau}\psi(x, \tau u)d\tau + \psi(x, u).$$

Since $\psi(x, u)$ is bounded for $|u| = 1$ then

$$|\psi(x, \lambda u)| \leq c_1 \int_\lambda^1 \frac{1}{\tau}e^{-\delta\tau}d\tau + c_1 \leq c_1 \left(\int_\lambda^1 \frac{1}{\tau}d\tau + 1\right) = c_1(1 + |\log\lambda|).$$

In the second case we begin with

$$\psi(x, \lambda u) = -\int_\lambda^\infty \frac{\partial}{\partial \tau}\psi(x, \tau u)d\tau$$

because $\psi(x, \lambda u) \to 0$ as $\lambda \to \infty$. Hence

$$|\psi(x, \lambda u)| \leq c_1 \int_\lambda^\infty \frac{1}{\lambda}e^{-\delta\tau}d\tau \leq c_1 \int_\lambda^\infty e^{-\delta\tau}d\tau = c_1' e^{-\delta\lambda}$$

for $\lambda \geq 1$. That's why

$$|\psi(x, \lambda u)| \leq c_1(1 + |\log\lambda|)e^{-\delta\lambda}, \quad \lambda > 0.$$

This proves 2).

3) Since $m\nu - l > n$ then the claim follows from above considerations immediately without differentiation.

□

Lemma 5. *Let $z \notin \mathbb{Z}_\theta$. Define the function*

$$I_\nu(x, y, z) = F^{-1}\left(\frac{b(x, \xi)}{(a(x, \xi) + 1 - z)^\nu}\right)(x - y),$$

where $b(x, \xi)$ and $a(x, \xi)$ are as in lemma 4, $x \in \Omega_1$ and $y \in \Omega$. Then there exists $\delta > 0$ such that

1) $m\nu - l < n$,

$$|I_\nu(x, y, z)| \leq c_1|x - y|^{m\nu - l - n}e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}};$$

2) $m\nu - l = n$,

$$|I_\nu(x, y, z)| \leq c_1\left(1 + \left|\log\left(|x - y|(1 + |z|)^{\frac{1}{m}}\right)\right|\right)e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}};$$

3) $m\nu - l > n$,

$$|I_\nu(x,y,z)| \le c_1(1+|z|)^{-\frac{m\nu-l-n}{m}} e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}}.$$

Proof. We know from Lemma 1 that

$$a(x,\xi)+1-z \asymp a(x,\xi)+1+|z|.$$

This fact allows us to consider, without loss of generality, the function

$$\widetilde{I_\nu}(x,y,z) = F^{-1}\left(\frac{b(x,\xi)}{(a(x,\xi)+1+|z|)^\nu}\right)(x-y).$$

But it is not so difficult to check that

$$\widetilde{I_\nu}(x,y,z) = (1+|z|)^{\frac{n-m\nu+l}{m}}\psi(x,\lambda u),$$

where $\lambda = |x-y|(1+|z|)^{\frac{1}{m}}$ and $u = \frac{x-y}{|x-y|}$. Applying Lemma 4 implies now 1)-3). □

Corollary. *Let $\alpha > 0, z \notin \mathbb{Z}_\theta, x \in \Omega_1$ and $y \in \Omega$. Then*

1) $m\nu - l < n + |\alpha|$,

$$|D_x^\alpha I_\nu(x,y,z)| \le c_1|x-y|^{m\nu-l-|\alpha|-n} e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}};$$

2) $m\nu - l = n + |\alpha|$,

$$|D_x^\alpha I_\nu(x,y,z)| \le c_1\left(1+\left|\log\left(|x-y|(1+|z|)^{\frac{1}{m}}\right)\right|\right) e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}};$$

3) $m\nu - l > n + |\alpha|$,

$$|D_x^\alpha I_\nu(x,y,z)| \le c_1(1+|z|)^{\frac{n+l+|\alpha|-m\nu}{m}} e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}}.$$

Exercise 43. Prove Corollary.

Now we are in the position to prove the main theorem.

Theorem 1. *For any $\overline{\Omega_1} = \Omega_1 \subset \Omega$ there is $R_0 > 0$ such that for all $k \ge n, n \in \mathbb{N}$ and for all $|z| > R_0, z \notin \mathbb{Z}_\theta$ there exists a fundamantal solution $F(x,y,z), x,y \in \Omega_1$ of an operator $A(x,D) - zI$. This fundamental solution has the form*

$$F(x,y,z) = F_k(x,y,z) + \int_{\Omega_1} F_k(x,u,z)h_k(u,y,z)du,$$

where $F_k = \sum_{j=0}^k F^{-1}\varphi_j, h_k(x,y,z)$ exists and satisfies the estimate

$$|h_k(x,y,z)| \le c_1(1+|z|)^{\frac{n-k-1}{m}} e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}}.$$

Proof. As we already know $F(x,y,z)$ of such form is a fundamental solution of the operator $A(x,D)-zI$ if and only if the function $h_k(x,y,z)$ solves the equation

$$h_k(x,y,z)=H_k(x,y,z)+\int_{\Omega_1}H_k(x,u,z)h_k(u,y,z)du,$$

where

$$H_k(x,y,z)=F^{-1}\left((a+1-z)\varphi_{k+1}\right)(x-y)$$

with

$$(a(x,\xi)+1-z)\varphi_{k+1}(x,\xi,z)=\sum_\nu\frac{b_\nu(x,\xi)}{(a(x,\xi)+1-z)^\nu}$$

and

$$m\nu-l_\nu\geq k+1.$$

Since $k\geq n$ then $m\nu-l_\nu\geq n+1>n$ and, therefore, we can apply part 3) of Lemma 5 and obtain

$$|H_k(x,y,z)|\leq c_1\sum_\nu(1+|z|)^{\frac{n-(m\nu-l_\nu)}{m}}e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}}\leq c_1(1+|z|)^{\frac{n-k-1}{m}}e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}}.$$

Next, we look for $h_k(x,y,z)$ as the series of iterations of H_k i.e.

$$h_k(x,y,z):=\sum_{j=1}^\infty H_k^{(j)}(x,y,z),$$

where $H_k^{(1)}=H_k$ and

$$H_k^{(j)}(x,y,z)=\int_{\Omega_1}H_k(x,u,z)H_k^{(j-1)}(u,y,z)du,\quad j\geq 2.$$

It is clear that $h_k(x,y,z)$ of such form is a (formal) solution of the corresponding integral equation for h_k. It remains merely to prove that for $|z|>>1, z\notin\mathbb{Z}_\theta$ this series converges uniformly with respect to x and y from Ω_1. To this end, let us consider the estimates for iterations of H_k. Indeed,

$$\begin{aligned}|H_k^{(2)}(x,y,z)| &\leq \int_{\Omega_1}|H_k(x,u,z)||H_k^{(1)}(u,y,z)|du\\ &\leq c_1^2\left((1+|z|)^{\frac{n-k-1}{m}}\right)^2\int_{\Omega_1}e^{-\delta(|x-u|+|u-y|)(1+|z|)^{\frac{1}{m}}}du\\ &\leq c_1^2\left((1+|z|)^{\frac{n-k-1}{m}}\right)^2|\Omega_1|e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}}\end{aligned}$$

since $|x-u|+|u-y|\geq|x-y|$. Here and later $|\Omega_1|$ denotes the Lebesgue measure of Ω_1. By induction we get

$$|H_k^{(j)}(x,y,z)| \leq \left(c_1(1+|z|)^{\frac{n-k-1}{m}}\right)^j|\Omega_1|^{j-1}e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}}.$$

Hence

$$\sum_{j=1}^{\infty} |H_k^{(j)}| \le c_1(1+|z|)^{\frac{n-k-1}{m}} e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}} \sum_{j=1}^{\infty} \left(c_1|\Omega_1|(1+|z|)^{\frac{n-k-1}{m}}\right)^{j-1}.$$

If we choose z so that

$$c_1|\Omega_1|(1+|z|)^{\frac{n-k-1}{m}} < 1$$

or

$$|z| > R_0 := (c_1|\Omega_1|)^{\frac{m}{k+1-n}} - 1$$

then the series converges uniformly with respect to $x, y \in \Omega_1$. Therefore $h_k(x, y, z)$ is well-defined and $F(x, y, z)$ exists. □

Corollary 1. *The fundamental solution $F(x, y, z)$ satisfies the following estimates:*

1) $m < n$,

$$|F(x, y, z)| \le c_1|x-y|^{m-n} e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}};$$

2) $m = n$,

$$|F(x, y, z)| \le c_1 \left(1 + \left|\log\left(|x-y|(1+|z|)^{\frac{1}{m}}\right)\right|\right) e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}};$$

3) $m > n$,

$$|F(x, y, z)| \le c_1(1+|z|)^{\frac{n-m}{m}} e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}},$$

where $x, y \in \Omega_1$ and $|z| > R_0, z \notin \mathbb{Z}_\theta$.

Proof. Since

$$F_k(x, y, z) = \sum_{j=0}^{k} F^{-1}\varphi_j(x-y) = \sum_{j=0}^{k}\sum_{\nu_j} F^{-1}\left(\frac{b_{\nu_j}}{(a+1-z)^{\nu_j+1}}\right)(x-y),$$

where $m\nu_j - l_j \ge j$ then we can apply Lemma 5 with $\nu = \nu_j + 1$ and $l = l_j$ for $j = 0, 1, \ldots, k$. That's why we can obtain

$$|F_k| \le c_1 e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}} \sum_{j=0}^{k}\sum_{\nu_j} \begin{cases} |x-y|^{m\nu_j+m-l_j-n}, & m\nu_j+m-l_j < n \\ 1 + \left|\log\left(|x-y|(1+|z|)^{\frac{1}{m}}\right)\right|, & m\nu_j+m-l_j = n \\ (1+|z|)^{\frac{l_j+n-m-m\nu_j}{m}}, & m\nu_j+m-l_j > n. \end{cases}$$

Let us remember that $\nu_0 = l_0 = 0$. That's why the first term of this sum can be estimated as

$$|I_0| \le c_1 e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}} \begin{cases} |x-y|^{m-n}, & m < n \\ 1 + \left|\log\left(|x-y|(1+|z|)^{\frac{1}{m}}\right)\right|, & m = n \\ (1+|z|)^{\frac{n-m}{m}}, & m > n. \end{cases}$$

But $m\nu_j - l_j \geq j$ means that the value $m\nu_j - l_j$ grows with respect to j. That's why the next terms in this sum have better estimates than I_0. This remark allows us to conclude that

$$|F_k(x,y,z)| \leq c_1 e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}} \begin{cases} |x-y|^{m-n}, & m<n \\ 1+\left|\log\left(|x-y|(1+|z|)^{\frac{1}{m}}\right)\right|, & m=n \\ (1+|z|)^{\frac{n-m}{m}}, & m>n \end{cases}$$

for $k = 0, 1, \ldots$. In order to obtain estimates for $F(x,y,z)$ it remains to investigate its second term, namely

$$\int_{\Omega_1} F_k(x,u,z)h_k(u,y,z)du.$$

Applying the estimates for F_k and h_k we obtain

$$\begin{aligned} \left|\int_{\Omega_1} F_k(x,u,z)h_k(u,y,z)du\right| &\leq c_1^2(1+|z|)^{\frac{n-k-1}{m}} \int_{\Omega_1} e^{-\delta(|x-u|+|u-y|)(1+|z|)^{\frac{1}{m}}} du \\ &\times \begin{cases} |x-u|^{m-n}, & m<n \\ 1+\left|\log\left(|x-u|(1+|z|)^{\frac{1}{m}}\right)\right|, & m=n \\ (1+|z|)^{\frac{n-m}{m}}, & m>n \end{cases} \\ &\leq c_1^2(1+|z|)^{\frac{n-k-1}{m}} e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}} \\ &\times \begin{cases} \int_{\Omega_1} |x-u|^{m-n}du, & m<n \\ \int_{\Omega_1}\left(1+\left|\log\left(|x-u|(1+|z|)^{\frac{1}{m}}\right)\right|\right)du, & m=n \\ (1+|z|)^{\frac{n-m}{m}}|\Omega_1|, & m>n. \end{cases} \end{aligned}$$

It is clear that

$$\int_{\Omega_1} |x-u|^{m-n}du \leq c_0$$

if $m<n$ and

$$(1+|z|)^{\frac{n-m}{m}}|\Omega_1| \leq c_0'$$

if $m>n$. If $m=n$ then

$$\begin{aligned} \int_{\Omega_1}\left(1+\left|\log\left(|x-u|(1+|z|)^{\frac{1}{m}}\right)\right|\right)du &\leq \int_{\Omega_1} du + \int_{\Omega_1} |\log|x-u||\,du \\ &+ \log(1+|z|)^{\frac{1}{m}} \int_{\Omega_1} du \\ &\leq c_1\left(1+\log(1+|z|)^{\frac{1}{m}}\right) \\ &\leq c_{1,\varepsilon}(1+|z|)^{\frac{\varepsilon}{m}}, \quad \varepsilon>0. \end{aligned}$$

Since $\varepsilon > 0$ and small enough then finally

$$\left|\int_{\Omega_1} F_k(x,u,z)h_k(u,y,z)du\right| \leq c_1 e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}}.$$

This finishes the proof. □

Corollary 2. *Let* $0 < |\alpha| < m, z \notin \mathbb{Z}_\theta, |z| > R_0$ *and* $x, y \in \Omega_1$. *Then*

1) $m < n + |\alpha|$,

$$|D_x^\alpha F(x,y,z)| \le c_1 |x-y|^{m-n-|\alpha|} e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}};$$

2) $m = n + |\alpha|$,

$$|D_x^\alpha F(x,y,z)| \le c_1 \left(1 + \left|\log\left(|x-y|(1+|z|)^{\frac{1}{m}}\right)\right|\right) e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}};$$

3) $m > n + |\alpha|$,

$$|D_x^\alpha F(x,y,z)| \le c_1 (1+|z|)^{\frac{n+|\alpha|-m}{m}} e^{-\delta|x-y|(1+|z|)^{\frac{1}{m}}}.$$

Exercise 44. Prove Corollary 2.

Exercise 45. Let $|\alpha| = m$ and $|x-y| \ge \varepsilon$. Prove that

$$|D_x^\alpha F(x,y,z)| \le c_\varepsilon e^{-\delta\varepsilon(1+|z|)^{\frac{1}{m}}}.$$

Let $\widehat{A}$ be an arbitrary self-adjoint extension of an elliptic differential operator $A(x, D)$ of even order m in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Without loss of generality we assume that $\widehat{A} \ge 0$. For $z \notin \mathbb{Z}_\theta$ we know that $(\widehat{A} - zI)^{-1}$ exists and is an integral operator with kernel $G(x,y,z)$ which is called the Green's function. We denote this inverse by $\widehat{G_z}$ i.e.

$$\widehat{G_z} f(x) = \int_\Omega G(x,y,z) f(y) dy.$$

Thus

$$(\widehat{A} - zI)\widehat{G_z} = \widehat{G_z}(\widehat{A} - zI) = I$$

and $\widehat{G_z} : L^2(\Omega) \to L^2(\Omega)$ is a bounded operator. We know also that for any $\overline{\Omega_1} = \Omega_1 \subset \Omega$ there is a fundamental solution $F(x,y,z)$ for $z \notin \mathbb{Z}_\theta$ and $|z| > R_0$. Moreover, we have the estimates for $D_x^\alpha F(x,y,z)$ for $|\alpha| \le m-1$, see Corollary for Theorem 1. Our main task now is to obtain estimates for $G(x,y,z)$.

Definition. A function $E(x,y,z), x, y \in \Omega, z \notin \mathbb{Z}_\theta$ is called a *parametrix* for $\widehat{A} - zI$ if the integral operator

$$\widehat{Q_z} := \widehat{G_z} - \widehat{E_z},$$

where $\widehat{E_z}$ is the integral operator with kernel $E(x,y,z)$, has the mapping property

$$\widehat{Q_z} : L^1(\Omega_1) \to \bigcap_{j=1}^{\infty} D(\widehat{A}^j)$$

for any $\overline{\Omega_1} = \Omega_1 \subset \Omega$.

Remark. It is easy to obtain the representation

$$(\widehat{A} - zI)\widehat{E_z} = I + \widehat{P_z},$$

where $\widehat{P_z} = -(\widehat{A} - zI)\widehat{Q_z}$ or $\widehat{Q_z} = -\widehat{G_z}\widehat{P_z}$.

If $\overline{\Omega_1} = \Omega_1 \subset \Omega$ then denote

$$\varepsilon_0 = \mathrm{dist}(\Omega_1, \partial\Omega) > 0.$$

For any fixed ε such that $0 < \varepsilon < \varepsilon_0$ define the function $\chi(x) \in C_0^\infty(\Omega)$ as

$$\chi(x) = \begin{cases} 1, & x \in \Omega_1^{\varepsilon/2} \\ 0, & x \in \Omega \setminus \Omega_1^{\varepsilon}, \end{cases}$$

where $\mathrm{dist}(\Omega_1, \partial\Omega_1^{\varepsilon/2}) = \mathrm{dist}(\Omega_1^{\varepsilon/2}, \partial\Omega_1^{\varepsilon}) = \varepsilon/2$ with $\Omega_1 \subset \Omega_1^{\varepsilon/2} \subset \Omega_1^{\varepsilon} \subset \Omega$ and $\mathrm{dist}(\Omega_1^{\varepsilon}, \partial\Omega) > 0$, see Figure 2.

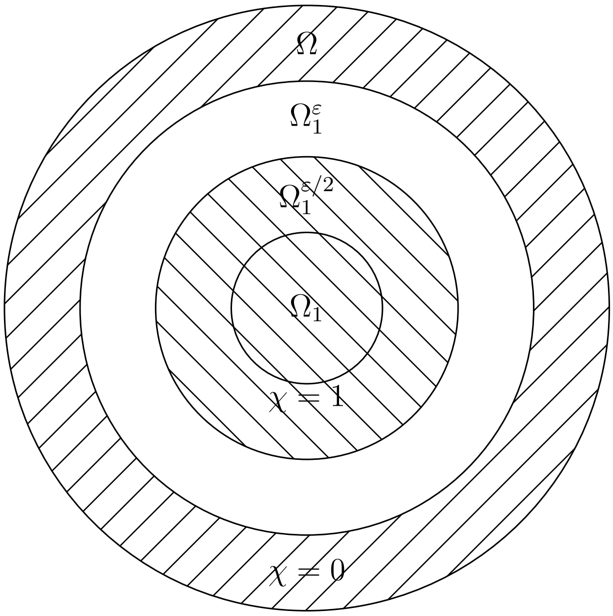

Figure 2:

Let us consider a fundamental solution $F(x, y, z)$ for $x, y \in \Omega_1^{\varepsilon}$. Define the function E by

$$E(x, y, z) := \chi(x)F(x, y, z)$$

which is well-defined for all $x \in \Omega$ and $y \in \Omega_1^{\varepsilon}$. Let us assume that $E(x, y, z) \equiv 0$ for $x \in \Omega$ and $y \in \Omega \setminus \Omega_1^{\varepsilon}$. Thus, $E(x, y, z)$ is defined for all $x, y \in \Omega$. It is also clear that $E(x, y, z) = F(x, y, z)$ for $x, y \in \Omega_1^{\varepsilon/2}$. The function $E(x, y, z)$ is called a *smoothed fundamental solution*.

Theorem 2. *The function $E(x,y,z)$ defined above is parametrix for $\widehat{A}-zI$. What is more,*

$$\left\| \widehat{A}^k \widehat{Q_z} f \right\|_{L^2(\Omega)} \le c_{k,\varepsilon}(1+|z|)^k e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}} \|f\|_{L^1(\Omega_1)}$$

for any $k \in \mathbb{N}$.

Proof. Since $E(x,y,z) \in C_0^\infty(\Omega)$ for $x \neq y$ then

$$\begin{aligned}(\widehat{A}-zI)E(x,y,z) &= (A(x,D)-zI)E(x,y,z) = \chi(x)(A(x,D)-zI)F(x,y,z) \\ &+ \sum_{\alpha>0} \frac{1}{\alpha!} D^\alpha \chi A^{(\alpha)}(x,D)F(x,y,z) = \delta(x-y) + P(x,y,z),\end{aligned}$$

where $x \in \Omega, y \in \Omega_1$ and $P(x,y,z)$ is defined by the second term in the sum. It is also easy to see that $D^\alpha \chi \neq 0$ only for $x \in \Omega_1^\varepsilon \setminus \Omega_1^{\varepsilon/2}$. Since $y \in \Omega_1$ then we may conclude that $P(x,y,z) \neq 0$ for all $x \in \Omega_1^\varepsilon \setminus \Omega_1^{\varepsilon/2}$ and $y \in \Omega_1$. This fact implies that $|x-y| \ge \varepsilon/2$. Next, since $\alpha > 0$ then the order of the differential operator $A^{(\alpha)}(x,D)$ is at most $m-1$. Hence we can apply the corollaries of Theorem 1 and obtain

$$|P(x,y,z)| \le c_\varepsilon e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}}.$$

Moreover, we prove that

$$|\widehat{A}^k P(x,y,z)| \le c_{\varepsilon,k}(1+|z|)^k e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}}$$

for any $k \in \mathbb{N}$. It suffices to prove this for $k=1$ and then use induction. Indeed, since $|x-y| \ge \varepsilon/2$ then $F(x,y,z)$ is a regular solution of the equation

$$(A(x,D)-zI)F(x,y,z) = 0.$$

In other words

$$\sum_{|\alpha|=m} a_\alpha(x) D_x^\alpha F = zF - \sum_{|\alpha|\le m-1} a_\alpha(x) D_x^\alpha F.$$

It follows from ellipticity that

$$\sum_{|\alpha|=m} a_\alpha(x) D_x^\alpha F \ge c_1 \sum_{|\alpha|=m} |D^\alpha F|.$$

Hence

$$\begin{aligned}\sum_{|\alpha|=m} |D^\alpha F| &\le c|z||F| + c\sum_{|\alpha|\le m-1} |D^\alpha F| \le c\left(|z|e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}} + e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}}\right) \\ &= c(1+|z|)e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}}\end{aligned}$$

by corollaries 1 and 2. Next, since $|x-y| \ge \varepsilon/2$ then

$$\begin{aligned}\widehat{A}P &\equiv A(x,D)P(x,y,z) = A(x,D)\sum_{\alpha>0} \frac{D^\alpha \chi}{\alpha!} A^{(\alpha)} F \\ &= \sum_{\alpha>0}\sum_{\beta\ge 0} \frac{1}{\alpha!}\frac{1}{\beta!} D^{\alpha+\beta}\chi A^{(\beta)}(x,D)\left(A^{(\alpha)}(x,D)F\right) = \sum_{\gamma>0} c_\gamma D^\gamma \chi B_\gamma(x,D)F,\end{aligned}$$

where $B_\gamma(x,D)$ is an operator of order $2m-|\gamma|\le 2m-1$. If $|\gamma|<m$ then the corresponding terms in this sum can be estimated from above as

$$c_\varepsilon e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}}.$$

If $|\gamma|\ge m$ then by ellipticity we obtain again that

$$B_\gamma(x,D)F \asymp D^\beta A(x,D)F$$

or

$$B_\gamma(x,D)F \asymp D^\beta(zF),$$

where $|\beta|=|\gamma|-m\le m-1$. Hence, we obtain in that case

$$|B_\gamma(x,D)F|\le c_\varepsilon|z|e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}}.$$

If we combine these two estimates then we obtain

$$|\widehat{A}P(x,y,z)|\le c_\varepsilon(1+|z|)e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}}$$

and by induction

$$|\,\widehat{A}^k\,P(x,y,z)|\le c_{\varepsilon,k}(1+|z|)^k e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}}.$$

Next we write

$$\widehat{A}\widehat{Q_z}=-\widehat{A}\widehat{G_z}\widehat{P_z}=-\left[\widehat{A}-zI+zI\right]\widehat{G_z}\widehat{P_z}=-\widehat{P_z}-z\widehat{G_z}\widehat{P_z}=-\widehat{P_z}+z\widehat{Q_z}.$$

This implies that

$$\left\|\widehat{A}\widehat{Q_z}f\right\|_{L^2}\le\left\|\widehat{P_z}f\right\|_{L^2}+|z|\left\|\widehat{Q_z}f\right\|_{L^2}.$$

But $\widehat{Q_z}=-\widehat{G_z}\widehat{P_z}$ and $\widehat{G_z}$ is a bounded operator in $L^2(\Omega)$. Hence

$$\left\|\widehat{A}\widehat{Q_z}f\right\|_{L^2(\Omega)}\le\left\|\widehat{P_z}f\right\|_{L^2(\Omega)}+c|z|\left\|\widehat{P_z}f\right\|_{L^2(\Omega)}.$$

So it remains to estimate $\left\|\widehat{P_z}f\right\|_{L^2(\Omega)}$. If $f\in L^1(\Omega_1)$ then

$$\begin{aligned}|\widehat{P_z}f(x)| &= \left|\int_{\Omega_1}P(x,y,z)f(y)dy\right|\le\int_{\Omega_1}|P(x,y,z)||f(y)|dy\\ &\le c_\varepsilon e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}}\|f\|_{L^1(\Omega_1)}.\end{aligned}$$

It follows that

$$\left\|\widehat{P_z}f\right\|_{L^2(\Omega)}\le c_\varepsilon e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}}\|f\|_{L^1(\Omega_1)}\left(\int_{\Omega_1}dx\right)^{1/2}=c_\varepsilon|\Omega_1|^{1/2}e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}}\|f\|_{L^1(\Omega_1)}.$$

Applying the identity

$$\widehat{A}^k\,\widehat{Q_z}=-\,\widehat{A}^{k-1}\,\widehat{P_z}+z\,\widehat{A}^{k-1}\,\widehat{Q_z}$$

we obtain

$$
\begin{aligned}
\left\| \widehat{A}^{k} \widehat{Q_z} f \right\|_{L^2(\Omega)} &\leq \left\| \widehat{A}^{k-1} \widehat{P_z} f \right\|_{L^2(\Omega)} + |z| \left\| \widehat{A}^{k-1} \widehat{Q_z} f \right\|_{L^2(\Omega)} \\
&\leq c_{\varepsilon,k}(1+|z|)^{k-1} e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}} \|f\|_{L^1(\Omega_1)} \\
&+ c_{\varepsilon,k}|z|(1+|z|)^{k-1} e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}} \|f\|_{L^1(\Omega_1)} \\
&\leq c_{\varepsilon,k}(1+|z|)^{k} e^{-\delta\frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}} \|f\|_{L^1(\Omega_1)} .
\end{aligned}
$$

This finishes the proof. □

9 Fractional powers of self-adjoint operators

If $\widehat{A} = \widehat{A}^*$ and $\widehat{A} > 0$, say $\widehat{A} \geq c_0 I, c_0 > 0$, then for $z \notin \mathbb{Z}_\theta$ we know that $(\widehat{A} - zI)^{-1} \equiv \widehat{G_z}$ is an integral operator with kernel $G(x, y, z)$ which can be represented in the form

$$G(x, y, z) = E(x, y, z) + Q(x, y, z),$$

where $E(x, y, z)$ is a smoothed fundamental solution satisfying the estimates

$$|D_x^\alpha E(x, y, z)| \leq c e^{-\delta |x-y|(1+|z|)^{\frac{1}{m}}} \begin{cases} |x-y|^{m-|\alpha|-n}, & m < n + |\alpha| \\ 1 + \left|\log\left(|x-y|(1+|z|)^{\frac{1}{m}}\right)\right|, & m = n + |\alpha| \\ (1+|z|)^{\frac{n+|\alpha|-m}{m}}, & m > n + |\alpha| \end{cases}$$

for $|\alpha| \leq m - 1$ and $x, y \in \Omega$, and $Q(x, y, z)$ is given by

$$Q(x, y, z) = -\int_\Omega G(x, u, z) P(u, y, z) du.$$

The integral operator $\widehat{Q_z}$ with this kernel has the mapping property

$$\widehat{Q_z} : L^1(\Omega_1) \to \Delta(\widehat{A}) \equiv \bigcap_{j=1}^{\infty} D(\widehat{A}^j),$$

and for each $k \in \mathbb{N}$

$$\left\| \widehat{A}^k \widehat{Q_z} f \right\|_{L^2(\Omega)} \leq c(1+|z|)^k e^{-\delta \frac{\varepsilon}{2}(1+|z|)^{\frac{1}{m}}} \|f\|_{L^1(\Omega_1)}.$$

Concerning the estimates for $Q(x, y, z)$ we have

$$|Q(x, y, z)| \leq c_1 e^{-\delta |x-y|(1+|z|)^{\frac{1}{m}}} \begin{cases} |x-y|^{m-n}, & m < n \\ 1 + \left|\log\left(|x-y|(1+|z|)^{\frac{1}{m}}\right)\right|, & m = n \\ (1+|z|)^{\frac{n-m}{m}}, & m > n \end{cases}$$

only for $x \in \Omega_1$ and $y \in \Omega$ because $G(x, y, z)$ is simply equal to the fundamental solution in that case. Next, for any $s > 0$ we can define $\widehat{A}^s$ by the spectral theorem as

$$\widehat{A}^s = \int_0^\infty \lambda^s dE_\lambda$$

with

$$D(\widehat{A}^s) = \left\{ f \in L^2(\Omega) : \int_0^\infty \lambda^{2s} d(E_\lambda f, f) < \infty \right\}.$$

Example 9.1. If $\widehat{A} = A_F > 0$ then

$$A_F^s f = \sum_{j=1}^{\infty} \lambda_j^s f_j e_j(x),$$

where $f_j = (f, e_j)_{L^2(\Omega)}, \{e_j\}_{j=1}^\infty$ is an orthonormal basis of eigenvectors and

$$D(A_F^s) = \left\{ f \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^{2s} |f_j|^2 < \infty \right\}.$$

Since we consider $\widehat{A} \geq c_0 I, c_0 > 0$ then we can also define

$$\widehat{A}^{-\tau} = \int_{c_0}^{\infty} \lambda^{-\tau} dE_\lambda, \quad \tau > 0,$$

with

$$D(\widehat{A}^{-\tau}) = \left\{ f \in L^2(\Omega) : \int_{c_0}^{\infty} \lambda^{-2\tau} d(E_\lambda f, f) < \infty \right\}.$$

Suppose that $0 < \tau < 1$. Then

$$\begin{aligned} \int_0^\infty \frac{t^{-\tau} dt}{\lambda + t} &\overset{y=t/\lambda}{=} \int_0^\infty \frac{\lambda^{-\tau} y^{-\tau}}{1+y} dy = \lambda^{-\tau} \int_0^\infty y^{-\tau}(1+y)^{-1} dy \\ &\overset{r=(1+y)^{-1}}{=} \lambda^{-\tau} \int_0^1 (1-r)^{-\tau} r^{\tau-1} dr = \lambda^{-\tau} B(\tau, 1-\tau) \\ &= \lambda^{-\tau} \frac{\Gamma(\tau)\Gamma(1-\tau)}{\Gamma(1)} = \lambda^{-\tau} \frac{\pi}{\sin \pi\tau} \end{aligned}$$

for any $\lambda > 0$. That's why we can rewrite the spectral representation for $\widehat{A}^{-\tau}, 0 < \tau < 1$ in the form

$$\begin{aligned} \widehat{A}^{-\tau} &= \int_{c_0}^{\infty} \lambda^{-\tau} dE_\lambda = \frac{\sin \tau\pi}{\pi} \int_{c_0}^{\infty} \left\{ \int_0^\infty \frac{t^{-\tau} dt}{\lambda + t} \right\} dE_\lambda \\ &= \frac{\sin \tau\pi}{\pi} \int_0^\infty t^{-\tau} \left\{ \int_{c_0}^{\infty} (\lambda + t)^{-1} dE_\lambda \right\} dt = \frac{\sin \tau\pi}{\pi} \int_0^\infty t^{-\tau} \widehat{G_{-t}} dt, \end{aligned}$$

because $z = -t \notin \mathbb{Z}_\theta$ and, therefore, $\widehat{G_{-t}}$ is well-defined.

Theorem 1. *If $0 < \tau < 1$ then $\widehat{A}^{-\tau}$ is the sum of the integral operators $\widehat{E}(\tau)$ and $\widehat{Q}(\tau)$ whose kernels are*

$$E_\tau(x, y) = \frac{\sin \tau\pi}{\pi} \int_0^\infty t^{-\tau} E(x, y, -t) dt$$

and

$$Q_\tau(x, y) = \frac{\sin \tau\pi}{\pi} \int_0^\infty t^{-\tau} Q(x, y, -t) dt,$$

respectively. Moreover,

$$|D_x^\alpha E_\tau(x, y)| \leq c \begin{cases} |x-y|^{m\tau - |\alpha| - n}, & m\tau < n + |\alpha| \\ 1 + |\log|x-y||, & m\tau = n + |\alpha| \\ 1, & m\tau > n + |\alpha| \end{cases}$$

for $|\alpha| \leq m-1$ and $x, y \in \Omega$ and $\widehat{Q}(\tau)$ has the mapping property

$$\widehat{Q}(\tau) : L^1(\Omega_1) \to \Delta(\widehat{A}).$$

Proof. The first part of this theorem follows immediately from the fact that

$$\widehat{G_{-t}} = \widehat{E_{-t}} + \widehat{Q_{-t}}$$

since

$$G(x, y, -t) = E(x, y, -t) + Q(x, y, -t).$$

Let us obtain now the estimates for $D_x^\alpha E_\tau(x, y), |\alpha| \leq m - 1$. Since

$$|D_x^\alpha E_\tau(x, y)| \leq \frac{\sin \tau\pi}{\pi} \int_0^\infty t^{-\tau} |D_x^\alpha E(x, y, -t)| dt$$

we make use of the estimates for $|D_x^\alpha E|$.

1) Let $m < n + |\alpha|$. Then $m\tau < n + |\alpha|$ also and thus

$$\begin{aligned}
|D_x^\alpha E_\tau(x, y)| &\leq c|x - y|^{m-|\alpha|-n} \int_0^\infty t^{-\tau} e^{-\delta|x-y|(1+t)^{\frac{1}{m}}} dt \\
&\leq c|x - y|^{m-|\alpha|-n} \int_0^\infty t^{-\tau} e^{-\delta|x-y|t^{\frac{1}{m}}} dt \\
&\left(\quad u := |x - y|t^{\frac{1}{m}}; t = \frac{u^m}{|x - y|^m}; dt = \frac{mu^{m-1}}{|x - y|^m} du \right) \\
&= c|x - y|^{m\tau-|\alpha|-n} \int_0^\infty u^{m(1-\tau)-1} e^{-\delta u} du \\
&= c|x - y|^{m\tau-|\alpha|-n},
\end{aligned}$$

because the last integral converges.

2) Let $m = n + |\alpha|$. Then again $m\tau < n + |\alpha|$ and

$$\begin{aligned}
|D_x^\alpha E_\tau(x, y)| &\leq c \int_0^\infty t^{-\tau} \left(1 + \left|\log\left(|x - y|(1 + t)^{\frac{1}{m}}\right)\right|\right) e^{-\delta|x-y|(1+t)^{\frac{1}{m}}} dt \\
&\leq c \int_0^\infty t^{-\tau} \left(1 + \left|\log\left(|x - y|t^{\frac{1}{m}}\right)\right|\right) e^{-\delta|x-y|t^{\frac{1}{m}}} dt \\
&\left(\quad u := |x - y|t^{\frac{1}{m}}; t = \frac{u^m}{|x - y|^m}; dt = \frac{mu^{m-1}}{|x - y|^m} du \right) \\
&= c|x - y|^{m\tau-m} \int_0^\infty u^{m(1-\tau)-1} (1 + |\log u|) e^{-\delta u} du \\
&= c|x - y|^{m\tau-m} = c|x - y|^{m\tau-|\alpha|-n}
\end{aligned}$$

because the last integral converges again.

3) In the case $m > n + |\alpha|$ our task is to estimate

$$|D_x^\alpha E_\tau(x, y)| \leq c \int_0^\infty t^{-\tau} (1 + t)^{-1+\frac{n+|\alpha|}{m}} e^{-\delta|x-y|t^{\frac{1}{m}}} dt$$

in three subcases.

a) Let $m\tau < n+|\alpha|$. Then $\tau + 1 - \frac{n+|\alpha|}{m} < 1$. Let us change the variables $u := |x-y|t^{\frac{1}{m}}$. Then the integral in question equals

$$c|x-y|^{m\tau - m}\int_0^\infty u^{m(1-\tau)-1}\left(1+\left(\frac{u}{|x-y|}\right)^m\right)^{\frac{n+|\alpha|-m}{m}} e^{-\delta u}du$$

or

$$c|x-y|^{m\tau-|\alpha|-n}\int_0^\infty u^{m(1-\tau)-1}\left(|x-y|^m + u^m\right)^{\frac{n+|\alpha|-m}{m}} e^{-\delta u}du.$$

It can be estimated from above by

$$c|x-y|^{m\tau-|\alpha|-n}\int_0^\infty u^{m(1-\tau)-1}u^{n+|\alpha|-m}e^{-\delta u}du$$

which equals

$$c|x-y|^{m\tau-|\alpha|-n}\int_0^\infty u^{n+|\alpha|-m\tau-1}e^{-\delta u}du$$

or

$$c|x-y|^{m\tau-|\alpha|-n}$$

because the last integral converges in this case ($n+|\alpha|-m\tau-1 > -1$).

b) Let $m\tau = n+|\alpha|$. Then $\tau + \left(1-\frac{n+|\alpha|}{m}\right) = 1$ and we consider the integral

$$c|x-y|^{m\tau-|\alpha|-n}\int_0^\infty u^{m(1-\tau)-1}\left(|x-y|^m + u^m\right)^{\frac{n+|\alpha|-m}{m}} e^{-\delta u}du$$

in two parts:

$$\int_0^{|x-y|} du + \int_{|x-y|}^\infty du := I_1 + I_2.$$

We have

$$\begin{aligned} I_1 &\le c\int_0^{|x-y|} u^{m(1-\tau)-1}|x-y|^{n+|\alpha|-m}du = c|x-y|^{n+|\alpha|-m}|x-y|^{m(1-\tau)} \\ &= c|x-y|^{n+|\alpha|-m\tau} = c \end{aligned}$$

and

$$\begin{aligned} I_2 &\le c\int_{|x-y|}^\infty u^{m(1-\tau)-1}u^{n+|\alpha|-m}e^{-\delta u}du = c\int_{|x-y|}^\infty u^{-1}e^{-\delta u}du \\ &\le c\left(1+|\log|x-y||\right). \end{aligned}$$

c) Let $m\tau > n+|\alpha|$. Then $\tau + \left(1-\frac{n+|\alpha|}{m}\right) > 1$ and

$$\begin{aligned} |D_x^\alpha E_\tau(x,y)| &\le c\int_0^\infty t^{-\tau}(1+t)^{-1+\frac{n+|\alpha|}{m}} e^{-\delta|x-y|t^{\frac{1}{m}}} dt \\ &\le c\int_0^\infty t^{-\tau}(1+t)^{-1+\frac{n+|\alpha|}{m}} dt = c \end{aligned}$$

because the last integral converges.

Hence the second part of this theorem is proved. Finally, using Theorem 2 of section 8 we can obtain

$$\begin{aligned}\left\|\widehat{A}^k \widehat{Q}(\tau)f\right\|_{L^2(\Omega)} &\leq \int_0^\infty t^{-\tau}\left\|\widehat{A}^k \widehat{Q_{-t}f}\right\|_{L^2(\Omega)} dt\\ &\leq c\int_0^\infty t^{-\tau}(1+t)^k e^{-\delta\frac{\varepsilon}{2}(1+t)^{\frac{1}{m}}}\left\|f\right\|_{L^1(\Omega_1)} dt \leq c\left\|f\right\|_{L^1(\Omega_1)},\end{aligned}$$

for any $k = 0, 1, \ldots$. This finishes the proof. □

As it was proved, $\widehat{A}^{-\tau}, 0 < \tau < 1$ is an integral operator with the kernel

$$K_\tau(x, y) \equiv E_\tau(x, y) + Q_\tau(x, y)$$

such that

$$|E_\tau(x,y)| \leq c\begin{cases}|x-y|^{m\tau-n}, & m\tau < n\\ 1 + |\log|x-y||, & m\tau = n\\ 1, & m\tau > n,\end{cases}$$

where $x, y \in \Omega$. But $E_\tau(x, y) \equiv 0$ for $x \in \Omega \setminus \Omega_1, y \in \Omega$ and for $x \in \Omega, y \in \Omega \setminus \Omega_1$. The integral operator $\widehat{Q}(\tau)$ with the kernel $Q_\tau(x, y)$ maps from $L^1(\Omega_1)$ to $\Delta(\widehat{A})$. But if we take into account $Q_\tau(x, y)$ we can get only the estimate

$$|K_\tau(x,y)| \leq c_1\begin{cases}|x-y|^{m\tau-n}, & m\tau < n\\ 1 + |\log|x-y||, & m\tau = n\\ 1, & m\tau > n,\end{cases}$$

where $x \in \Omega_1, y \in \Omega$ and $c_1 = c(\Omega_1)$. If now $\tau \geq 1$ then we use the fact that

$$\widehat{A}^{-\tau} = \widehat{A}^{-\tau_1} \circ \widehat{A}^{-\tau_2} \circ \cdots \circ \widehat{A}^{-\tau_l},$$

where $\tau_1 + \tau_2 + \cdots + \tau_l = \tau$ with $0 < \tau_j < 1, j = 1, 2, \ldots, l$. Due to this fact we may conclude that $\widehat{A}^{-\tau}$ is an integral operator with the kernel $K_\tau(x, y)$ which can be calculated as

$$K_\tau(x,y) = \int_\Omega K_{\tau_1}(x,u_1)du_1\int_\Omega K_{\tau_2}(u_1,u_2)du_2\cdots\int_\Omega K_{\tau_{l-1}}(u_{l-2},u_{l-1})K_{\tau_l}(u_{l-1},y)du_{l-1}.$$

Lemma. *Let $0 < \tau_1 < 1$ and $0 < \tau_2 < 1$. Then*

$$|K_{\tau_1+\tau_2}(x,y)| \leq c_1\begin{cases}|x-y|^{m(\tau_1+\tau_2)-n}, & m\tau_1 + m\tau_2 < n\\ 1 + |\log|x-y||, & m\tau_1 + m\tau_2 = n\\ 1, & m\tau_1 + m\tau_2 > n,\end{cases}$$

where $x, y \in \Omega_1$ and $c_1 = c(\Omega_1)$.

Exercise 46. If $0 < \alpha_1 < n, 0 < \alpha_2 < n$ and $\alpha_1 + \alpha_2 > n$ then

$$\int_\Omega |x-u|^{-\alpha_1}|u-y|^{-\alpha_2}du \le \int_{\mathbb{R}^n} |x-u|^{-\alpha_1}|u-y|^{-\alpha_2}du \le c_0|x-y|^{n-(\alpha_1+\alpha_2)}.$$

Exercise 47. Let Ω be bounded. If $0 < \alpha_1 < n, 0 < \alpha_2 < n$ and $\alpha_1 + \alpha_2 = n$ then

$$\int_\Omega |x-u|^{-\alpha_1}|u-y|^{-\alpha_2}du \le c(\Omega)\left(1 + |\log|x-y||\right).$$

Exercise 48. Let Ω be bounded. If $0 < \alpha_1 < n, 0 < \alpha_2 < n$ and $\alpha_1 + \alpha_2 < n$ then

$$\int_\Omega |x-u|^{-\alpha_1}|u-y|^{-\alpha_2}du \le c(\Omega).$$

Proof of lemma. Since

$$|K_{\tau_1+\tau_2}(x,y)| \le \int_{\Omega_1} |K_{\tau_1}(x,u)||K_{\tau_2}(u,y)|du$$

then we can apply the estimates for K_{τ_1} and K_{τ_2} with $0 < \tau_1 < 1$, $0 < \tau_2 < 1$, $\alpha_1 = n - m\tau_1$ and $\alpha_2 = n - m\tau_2$.

1) Let $m\tau_1 < n, m\tau_2 < n$ and $m(\tau_1+\tau_2) < n$. Then $0 < \alpha_1 < n, 0 < \alpha_2 < n$ and $\alpha_1 + \alpha_2 > n$. These conditions imply that

$$\begin{aligned} |K_{\tau_1+\tau_2}(x,y)| &\le c_1 \int_{\Omega_1} |x-u|^{m\tau_1-n}|u-y|^{m\tau_2-n}du \\ &\le c_1|x-y|^{n-(n-m\tau_1)-(n-m\tau_2)} = c_1|x-y|^{m(\tau_1+\tau_2)-n}. \end{aligned}$$

2) Let $m\tau_1 < n, m\tau_2 < n$ and $m(\tau_1+\tau_2) = n$. Then $\alpha_1 + \alpha_2 = n$ and thus

$$|K_{\tau_1+\tau_2}(x,y)| \le c_1\left(1 + |\log|x-y||\right).$$

3) Let $m\tau_1 < n, m\tau_2 < n$ and $m(\tau_1+\tau_2) > n$. Then $\alpha_1 + \alpha_2 < n$. That's why

$$|K_{\tau_1+\tau_2}(x,y)| \le c_1.$$

4) Let $m\tau_1 < n, m\tau_2 = n$. Then $m(\tau_1+\tau_2) > n$ and we obtain

$$\begin{aligned} |K_{\tau_1+\tau_2}(x,y)| &\le c_1 \int_{\Omega_1} |x-u|^{m\tau_1-n}\left(1 + |\log|u-y||\right)du \\ &\le c_\varepsilon \int_{\Omega_1} |x-u|^{m\tau_1-n}|u-y|^{-\varepsilon}du \le c_\varepsilon \end{aligned}$$

if we choose $0 < \varepsilon < m\tau_1$.

The remaining five cases can be considered in a similar manner. □

Remark. It follows by induction that

$$|K_\tau(x,y)| \le c_1 \begin{cases} |x-y|^{m\tau-n}, & m\tau < n \\ 1 + |\log|x-y||\,, & m\tau = n \\ 1, & m\tau > n, \end{cases}$$

where $x, y \in \Omega_1, c_1 = c(\Omega_1)$ and $\tau > 0$. These estimates can be extended to hold for $x \in \Omega_1$ and $y \in \Omega$.

Theorem 2. *Suppose that $\tau > \frac{n}{2m}$. Then*

$$\left\| \widehat{A}^{-\tau} f \right\|_{L^\infty(\Omega_1)} \le c_1 \|f\|_{L^2(\Omega)},$$

where $\overline{\Omega_1} = \Omega_1 \subset \Omega$ and $c_1 = c(\Omega_1)$.

Proof. For $f \in L^2(\Omega)$ and $\tau > 0$ it follows from lemma above and Cauchy-Schwarz-Bunjakovskii inequality that

$$\begin{aligned} \left\| \widehat{A}^{-\tau} f \right\|_{L^\infty(\Omega_1)} &\le \sup_{x\in\Omega_1} \int_\Omega |K_\tau(x,y)||f(y)|dy \\ &\le c_1 \sup_{x\in\Omega_1} \int_\Omega |f(y)| \begin{cases} |x-y|^{m\tau-n}, & m\tau < n \\ 1 + |\log|x-y||\,, & m\tau = n \\ 1, & m\tau > n, \end{cases} dy \\ &\le c_1 \sup_{x\in\Omega_1} \|f\|_{L^2(\Omega)} \left(\int_\Omega \begin{cases} |x-y|^{2m\tau-2n}, & m\tau < n \\ (1 + |\log|x-y||)^2, & m\tau = n \\ 1, & m\tau > n, \end{cases} dy \right)^{1/2} \\ &\le c_1 \|f\|_{L^2(\Omega)} \end{aligned}$$

because $\tau > \frac{n}{2m}$ if and only if $2m\tau - 2n > -n$ which makes the latter integral converge. □

Now let us assume for simplicity that $\widehat{A} = A_F > 0$ is the Friedrichs extension. In that case

$$A_F f = \sum_{j=1}^\infty \lambda_j f_j e_j(x),$$

where $\lambda_j > 0, \lambda_j \to \infty$ and f_j are the Fourier coefficients of $f \in L^2(\Omega)$ with respect to $\{e_j(x)\}_{j=1}^\infty$. The orthonormal system of eigenvectors $\{e_j(x)\}_{j=1}^\infty$ forms an orthonormal basis in $L^2(\Omega)$. By the spectral theorem we know also that

$$D(A_F^s) = \left\{ f \in L^2(\Omega) : \sum_{j=1}^\infty \lambda_j^{2s} |f_j|^2 < \infty \right\}.$$

Corollary. *Let $\Omega_1 \subset \Omega$ be a compact set and let $\tau > \frac{n}{2m}$. Then*

$$\sum_{j=1}^{\infty} \lambda_j^{-2\tau} |e_j(x)|^2 \leq c_1$$

holds uniformly with respect to $x \in \Omega_1$ with $c_1 = c(\Omega_1)$.

Proof. Since

$$A_F^{-\tau} f = \sum_{j=1}^{\infty} \lambda_j^{-\tau} f_j e_j(x)$$

then Theorem 2 implies that

$$\left| \sum_{j=1}^{\infty} \lambda_j^{-\tau} f_j e_j(x) \right| \leq c_1 \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2}, \quad x \in \Omega_1.$$

The left hand side of this inequality can be considered as the inner product of

$$\vec{e} = (\lambda_1^{-\tau} e_1(x), \ldots, \lambda_j^{-\tau} e_j(x), \ldots)$$

and

$$\vec{f} = (\overline{f_1}, \ldots, \overline{f_j}, \ldots)$$

in $l^2(\mathbb{C})$. Hence,

$$\left| (\vec{e}, \vec{f})_{l^2} \right| \leq c_1 \left\| \vec{f} \right\|_{l^2}.$$

By duality we may conclude that

$$\|\vec{e}\|_{l^2} \leq c_1$$

or

$$\sum_{j=1}^{\infty} \lambda_j^{-2\tau} |e_j(x)|^2 \leq c_1^2.$$

□

Theorem 3. *The Fourier series*

$$\sum_{j=1}^{\infty} f_j e_j(x)$$

converges absolutely and uniformly on each compact set $\overline{\Omega_1} \subset \Omega$ for each function f from $D(A_F^{\tau})$ with $\tau > \frac{n}{2m}$.

Proof. Using the corollary above and Cauchy-Schwarz-Bunjakovskii inequality we obtain

$$\sum_{j=1}^{\infty} |f_j||e_j(x)| \leq \left(\sum_{j=1}^{\infty} \lambda_j^{2\tau} |f_j|^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} \lambda_j^{-2\tau} |e_j(x)|^2 \right)^{1/2} \leq c_1 \left(\sum_{j=1}^{\infty} \lambda_j^{2\tau} |f_j|^2 \right)^{1/2},$$

where $\tau > \frac{n}{2m}$. But the latter number series converges since $f \in D(A_F^{\tau})$. □

Remark. We know that

$$\overset{\circ}{W}{}_2^m(\Omega) \subset D(A_F).$$

Then for $\tau \in \mathbb{N}$ we may conclude that

$$\overset{\circ}{W}{}_2^{m\tau}(\Omega) \subset D(A_F^\tau).$$

This embedding implies that if $\tau > \frac{n}{2m}$ or $m\tau > n/2$ then the Fourier series corresponding to $f \in \overset{\circ}{W}{}_2^{m\tau}(\Omega)$ converges absolutely. Our aim is to prove this fact for any τ such that $m\tau > n/2$.

Let A be a non-negative self-adjoint operator in a Hilbert space H. Due to spectral theorem we can characterize $D(A)$ as follows: $f \in D(A)$ if and only if

$$\int_0^\infty (1+\lambda^2) d(E_\lambda f, f) < \infty$$

and define a new norm

$$\|f\|_{D(A)} := \|f\|_H + \|Af\|_H .$$

Definition. Let $\{G(t)\}_{t>0}$ be a family of bounded linear operators from H to H. This family is called an *equi-bounded, strongly continuous semi-group* if

1) $G(t+s)f = G(t)(G(s)f)$ for $s, t > 0$ and $f \in H$.

2) $\|G(t)f\|_H \le M \|f\|_H$ for $t > 0$ and $f \in H$ with $M > 0$ which does not depend on t or f.

3) $\lim_{t\to+0} \|G(t)f - f\|_H = 0$ for $f \in H$.

Remark. We can complete this definition by $G(0) := I$.

Definition. The *infinitesimal generator* A of the semi-group $\{G(t)\}_{t>0}$ is defined by the formula

$$\lim_{t\to 0} \left\| \frac{G(t)-I}{t} f - Af \right\|_H = 0$$

with domain $D(A)$ consisting of all $f \in H$ such that

$$\lim_{t\to 0} \frac{G(t)-I}{t} f$$

exists in H.

Remark. In the sense of the previous definition we write $G'(0) = A$.

Example 9.2. Let $H = L^2(\mathbb{R}^n)$. Let $\omega(\xi)$ be an infinitely differentiable positive function on $\mathbb{R}^n \setminus \{0\}$, which is positively homogeneous of order $m > 0$ i.e. $\omega(t\xi) = |t|^m \omega(\xi)$. Let us define the family $\{G(t)\}_{t>0}$ by the formula

$$G(t)f := F^{-1}\left(e^{-t\omega(\xi)} Ff\right), \quad f \in L^2(\mathbb{R}^n).$$

It is clear that $G(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Moreover, 1)

$$G(t+s)f = F^{-1}\left(e^{-(t+s)\omega(\xi)}Ff\right) = F^{-1}\left(e^{-t\omega(\xi)}FF^{-1}\left(e^{-s\omega(\xi)}Ff\right)\right) = G(t)(G(s)f).$$

2)

$$\|G(t)f\|_{L^2} = \left\|F^{-1}\left(e^{-t\omega(\xi)}Ff\right)\right\|_{L^2} = \left\|e^{-t\omega(\xi)}Ff\right\|_{L^2} \le \|Ff\|_{L^2} = \|f\|_{L^2}.$$

3)

$$\begin{aligned}\|G(t)f - f\|_{L^2} &= \left\|F^{-1}\left(e^{-t\omega(\xi)}Ff - Ff\right)\right\|_{L^2} = \left\|\left(e^{-t\omega(\xi)} - 1\right)Ff\right\|_{L^2} \\ &\to \|Ff\|_{L^2} = \|f\|_{L^2}, \quad t \to 0\end{aligned}$$

by the Lebesgue theorem. Also by Lebesgue theorem we have

$$\lim_{t\to 0}\frac{G(t)f - f}{t} \overset{H}{=} \lim_{t\to 0} F^{-1}\left(\frac{e^{-t\omega(\xi)} - 1}{t}Ff\right) = -F^{-1}\left(\omega(\xi)Ff\right) \equiv Af.$$

The domain of A is

$$D(A) = \left\{f \in L^2 : \|\omega(\xi)Ff\|_{L^2} < \infty\right\}.$$

For instance, if $\omega(\xi) = |\xi|^2$ then $A \equiv \Delta$ and $D(A) = W_2^2(\mathbb{R}^n)$.

Example 9.3. Let $A = A^* \ge 0$. Define

$$G(t) := e^{itA} \equiv \int_0^\infty e^{it\lambda} dE_\lambda.$$

Then

1)

$$G(t+s) = \int_0^\infty e^{i(t+s)\lambda} dE_\lambda = \int_0^\infty e^{it\lambda}e^{is\lambda} dE_\lambda = \int_0^\infty e^{it\lambda} dE_\lambda \int_0^\infty e^{is\mu} dE_\mu = G(t)G(s).$$

2)

$$\|G(t)f\|^2 = \int_0^\infty |e^{it\lambda}|^2 d(E_\lambda f, f) = \|f\|^2.$$

3)

$$\|G(t)f - f\|^2 = \int_0^\infty |e^{it\lambda} - 1|^2 d(E_\lambda f, f) \to 0, \quad t \to 0.$$

and

$$\frac{G(t)f - f}{t} = \int_0^\infty \frac{e^{it\lambda} - 1}{t} dE_\lambda f \to i\int_0^\infty \lambda dE_\lambda f \equiv iA, \quad t \to 0.$$

We will consider now J. Peetre's method of real interpolation or the K-method. We have $A = A^* \geq 0$ in a Hilbert space H. For any $k \in \mathbb{N}$ denote by D^k the domain of A^k. We define the interpolation space $D^{\alpha,p}$, where $\alpha > 0$ and $1 \leq p \leq \infty$ as follows: Set

$$K(t,f) := \inf_{f=f_0+f_1} \left(\|f_0\|_H + t\,\|f_1\|_{D^k} \right), \quad f \in H, 0 < t < \infty,$$

where $f_0 \in H$ and $f_1 \in D^k$. This functional is called the *functional of Peetre* or the *K-functional*. Then, if $0 < \alpha < k$ the space $D^{\alpha,p}$ is the space of all $f \in H$ such that

$$\left(\int_0^\infty \left(t^{-\frac{\alpha}{k}} K(t,f) \right)^p \frac{dt}{t} \right)^{1/p} < \infty$$

with the norm

$$\|f\|_{D^{\alpha,p}} := \left(\int_0^\infty \left(t^{-\frac{\alpha}{k}} K(t,f) \right)^p \frac{dt}{t} \right)^{1/p} < \infty.$$

We shall denote also

$$(H, D^k)_{\frac{\alpha}{k},p} := D^{\alpha,p}.$$

Remark. If $p \leq q$ then

$$D^{\alpha,p} \subset D^{\alpha,q}$$

and if $\alpha > \beta$ then

$$D^{\alpha,p} \subset D^{\beta,q}$$

for any p and q.

Theorem 4. *Let $G(t)$ be an equi-bounded, strongly continuous semi-group on H with infinitesimal generator A. Then*

1)

$$K(t,f) \asymp h(t,f) + \min(1,t)\,\|f\|_H\,,$$

where $h(t,f) = \sup_{s<t} \|G(s)f - f\|_H$.

2)

$$\|f\|_{(H,D(A))_{\theta,p}} \asymp \|f\|_H + \left(\int_0^\infty \left(t^{-\theta} h(t,f) \right)^p \frac{dt}{t} \right)^{1/p},$$

where $0 < \theta < 1$ and $1 \leq p \leq \infty$.

Proof. Let us assume that $f = f_0 + f_1$, where $f_0 \in H$ and $f_1 \in D(A)$. Then

$$\begin{aligned} h(t,f) &= \sup_{s<t} \|G(s)f - f\|_H \leq \sup_{s<t} \|G(s)f_0 - f_0\|_H + \sup_{s<t} \|G(s)f_1 - f_1\|_H \\ &\leq (M+1)\,\|f_0\|_H + \sup_{s<t} \|G(s)f_1 - f_1\|_H\,. \end{aligned}$$

Since $G'(0) = A$ then $G'(t) = AG(t) = G(t)A$. Indeed,

$$\begin{aligned} G'(t) &= \lim_{\tau \to 0} \frac{G(t+\tau) - G(t)}{\tau} = G(t) \lim_{\tau \to 0} \frac{G(\tau) - I}{\tau} \\ &= \lim_{\tau \to 0} \frac{G(\tau) - I}{\tau} G(t) = G(t)A = AG(t). \end{aligned}$$

We can thus write

$$G(t)f - f = \int_0^t G(s)Afds = \int_0^t AG(s)fds.$$

That's why

$$\begin{aligned}\sup_{s<t} \|G(s)f_1 - f_1\|_H &\le \int_0^t \sup_{s<t} \|G(s)Af_1\|_H ds \le M\int_0^t \|Af_1\|_H ds \\ &= Mt\|Af_1\|_H \le Mt\|f_1\|_{D(A)}.\end{aligned}$$

Therefore

$$h(t,f) \le (M+1)\|f_0\|_H + Mt\|f_1\|_{D(A)} \le (M+1)K(t,f).$$

Note that also $\min(1,t)\|f\|_H \le K(t,f)$. Indeed, if $0<t<1$ then

$$K(t,f) = \inf_{f=f_0+f_1}\left(\|f_0\|_H + t\|f_1\|_{D(A)}\right) \ge t\inf_{f=f_0+f_1}\left(\|f_0\|_H + \|f_1\|_{D(A)}\right) = t\|f\|_H.$$

If $t \ge 1$ then

$$K(t,f) \ge \inf_{f=f_0+f_1}\left(\|f_0\|_H + \|f_1\|_{D(A)}\right) = \|f\|_H.$$

Thus

$$h(t,f) + \min(1,t)\|f\|_H \le (M+1)K(t,f) + K(t,f) = (M+2)K(t,f).$$

For the other half of 1) we argue as follows. By the definition of infimum we have

$$K(t,f) \equiv \inf_{f=f_0+f_1}\left(\|f_0\|_H + t\|f_1\|_{D(A)}\right) \le \|f\|_H$$

under the choice $f_1 \equiv 0$ and $f = f_0$. Therefore

$$K(t,f) \le \|f\|_H \le h(t,f) + \|f\|_H = h(t,f) + \min(1,t)\|f\|_H$$

for $t \ge 1$. If $0<t<1$ then we put

$$f_1 = t^{-1}\int_0^t G(s)fds, \quad f_0 = f - f_1$$

for any $f \in H$. Then we have

$$\begin{aligned}K(t,f) &\le \|f_0\|_H + t\|f_1\|_{D(A)} = \left\|t^{-1}\int_0^t G(s)fds - f\right\|_H + t\left\|t^{-1}\int_0^t G(s)fds\right\|_{D(A)} \\ &= \left\|t^{-1}\int_0^t (G(s)f-f)ds\right\|_H + \left\|\int_0^t G(s)fds\right\|_{D(A)} \\ &\le \sup_{s<t}\|G(s)f-f\|_H + \left\|\int_0^t G(s)fds\right\|_H + \left\|\int_0^t AG(s)fds\right\|_H \\ &\le \sup_{s<t}\|G(s)f-f\|_H + Mt\|f\|_H + \|G(t)f-f\|_H \\ &\le h(t,f) + Mt\|f\|_H + h(t,f) \le \max(2,M)\left(h(t,f) + t\|f\|_H\right).\end{aligned}$$

This completes the proof of 1).

Since

$$\|f\|_{(H,D(A))_{\theta,p}} = \left(\int_0^\infty \left(t^{-\theta}K(t,f)\right)^p \frac{dt}{t}\right)^{1/p}$$

then part 1) implies that

$$\begin{aligned}
\|f\|_{(H,D(A))_{\theta,p}} &\asymp \left(\int_0^\infty \left(t^{-\theta}\min(1,t)\right)^p \frac{dt}{t}\right)^{1/p} \|f\|_H + \left(\int_0^\infty \left(t^{-\theta}h(t,f)\right)^p \frac{dt}{t}\right)^{1/p} \\
&= \left\{\int_0^1 t^{(1-\theta)p-1}dt + \int_1^\infty \frac{dt}{t^{1+\theta p}}\right\}^{1/p} \|f\|_H \\
&+ \left(\int_0^\infty \left(t^{-\theta}h(t,f)\right)^p \frac{dt}{t}\right)^{1/p} \\
&\asymp \|f\|_H + \left(\int_0^\infty \left(t^{-\theta}h(t,f)\right)^p \frac{dt}{t}\right)^{1/p}.
\end{aligned}$$

This completes the proof. □

Corollary. *Let A be a self-adjoint and non-negative operator in a Hilbert space H. Then*

$$(H, D(A^k))_{\theta,2} = D(A^{k\theta})$$

for any $k \in \mathbb{N}$ and $0 < \theta < 1$.

Proof. Due to Example 9.3 we may conclude that the family

$$G(t) := e^{itA^k} = \int_0^\infty e^{it\lambda^k} dE_\lambda, \quad t > 0$$

is an equi-bounded, strongly continuous semi-group with the infinitesimal generator iA^k. Since $D(iA^k) = D(A^k)$ and $(H, D(iA^k))_{\theta,2} = (H, D(A^k))_{\theta,2}$ then it follows from Theorem 4 that

$$\begin{aligned}
\|f\|_{(H,D(A^k))_{\theta,2}} &\asymp \|f\|_H + \left(\int_0^\infty t^{-2\theta}h(t,f)^2 \frac{dt}{t}\right)^{1/2} \\
&= \|f\|_H + \left(\int_0^\infty t^{-2\theta} \|G(t)f - f\|^2 \frac{dt}{t}\right)^{1/2} \\
&= \|f\|_H + \left(\int_0^\infty t^{-2\theta} \int_0^\infty |e^{it\lambda^k} - 1|^2 d\|E_\lambda f\|^2 \frac{dt}{t}\right)^{1/2} \\
&= \|f\|_H + \left(\int_0^\infty d\|E_\lambda f\|^2 \int_0^\infty \frac{|e^{it\lambda^k} - 1|^2}{t^{2\theta}} \frac{dt}{t}\right)^{1/2} \\
&\overset{\xi = t\lambda^k}{=} \|f\|_H + \left(\int_0^\infty \lambda^{2\theta k} d\|E_\lambda f\|^2 \int_0^\infty \frac{|e^{i\xi} - 1|^2}{\xi^{2\theta+1}} d\xi\right)^{1/2} \\
&= \|f\|_H + c_\theta \left(\int_0^\infty \lambda^{2\theta k} d\|E_\lambda f\|^2\right)^{1/2} = \|f\|_H + c_\theta \left\|A^{k\theta} f\right\|_H,
\end{aligned}$$

since the inner integral with respect to ξ converges for $0 < \theta < 1$. □

Remark. The corollary above allows us to conclude that

$$(H, D(A^k))_{\theta,1} \subset D(A^{k\theta}) \subset (H, D(A^k))_{\theta,\infty}$$

for any $k \in \mathbb{N}$ and $0 < \theta < 1$.

Let A_F be Friedrichs extension of an elliptic differential operator of order m with an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of eigenvectors in $L^2(\Omega)$, where Ω is bounded. Let also $A_F = A_F^* > 0$.

Theorem 5 (Peetre). *For any $f \in (L^2(\Omega), D(A^k))_{\frac{n}{2mk},1} \equiv D^{\frac{n}{2m},1}$ the series*

$$\sum_{j=1}^{\infty} f_j e_j(x),$$

where $f_j = (f, e_j)_{L^2(\Omega)}$ converges absolutely and uniformly in compact parts of Ω.

Proof. Since A_F has pure discrete spectrum $\{\lambda_j\}_{j=1}^{\infty}$ then

$$E_\lambda f(x) = \sum_{\lambda_j<\lambda} f_j e_j(x)$$

and, therefore,

$$\theta(x, y, \lambda) = \sum_{\lambda_j<\lambda} \overline{e_j(x)} e_j(y).$$

We will use Gårding's estimate

$$\theta(x, x, \lambda) = \sum_{\lambda_j<\lambda} |e_j(x)|^2 \le c_1 \lambda^{\frac{n}{m}},$$

where $c_1 = c(\Omega_1)$ can be chosen independent of $x \in \Omega_1, \overline{\Omega_1} = \Omega_1 \subset \Omega$. We have

$$\begin{aligned}
\sum_{j=1}^{\infty} |f_j e_j(x)| &\asymp \sum_{\nu=0}^{\infty} \sum_{2^\nu \le \lambda_j < 2^{\nu+1}} |f_j e_j(x)| \\
&\le \sum_{\nu=0}^{\infty} \left(\sum_{2^\nu \le \lambda_j < 2^{\nu+1}} |f_j|^2 \right)^{1/2} \left(\sum_{2^\nu \le \lambda_j < 2^{\nu+1}} |e_j(x)|^2 \right)^{1/2} \\
&\le c \sum_{\nu=0}^{\infty} 2^{\nu \frac{n}{2m}} \left(\sum_{2^\nu \le \lambda_j < 2^{\nu+1}} |f_j|^2 \right)^{1/2}.
\end{aligned}$$

It is clear that

$$\left(\sum_{2^\nu \le \lambda_j < 2^{\nu+1}} |f_j|^2 \right)^{1/2} \le \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} = \|f\|_{L^2}. \tag{9.1}$$

Since

$$f_j = (f, e_j) = \lambda_j^{-k}(f, \lambda_j^k e_j) = \lambda_j^{-k}(f, A_F^k e_j) = \lambda_j^{-k}(A_F^k f, e_j) = \lambda_j^{-k}(A_F^k f)_j$$

then we have also

$$\begin{aligned}\left(\sum_{2^\nu \le \lambda_j < 2^{\nu+1}} |f_j|^2\right)^{1/2} &= \left(\sum_{2^\nu \le \lambda_j < 2^{\nu+1}} \lambda_j^{-2k} |(A_F^k f)_j|^2\right)^{1/2} \\ &\le 2^{-k\nu} \left(\sum_{2^\nu \le \lambda_j < 2^{\nu+1}} |(A_F^k f)_j|^2\right)^{1/2} \le 2^{-k\nu} \|f\|_{D^k}. \quad (9.2)\end{aligned}$$

Applying (9.1) to f_0 and (9.2) to f_1 we get for $f = f_0 + f_1, f_0 \in L^2(\Omega), f_1 \in D(A_F^k)$ that

$$\left(\sum_{2^\nu \le \lambda_j < 2^{\nu+1}} |f_j|^2\right)^{1/2} \le \|f_0\|_{L^2} + 2^{-k\nu} \|f_1\|_{D^k}.$$

It follows that

$$\left(\sum_{2^\nu \le \lambda_j < 2^{\nu+1}} |f_j|^2\right)^{1/2} \le K(2^{-\nu k}, f).$$

That's why we obtain

$$\sum_{j=1}^{\infty} |f_j e_j(x)| \le c_1 \sum_{\nu=0}^{\infty} 2^{\frac{\nu n}{2m}} K(2^{-\nu k}, f) \le c_1 \sum_{\nu=-\infty}^{\infty} 2^{\frac{\nu n}{2m}} K(2^{-\nu k}, f).$$

Next we are going to prove that

$$\sum_{\nu=-\infty}^{\infty} 2^{\frac{\nu n}{2m}} K(2^{-\nu k}, f) \le c \int_0^\infty t^{-\frac{n}{2mk}} K(t, f) \frac{dt}{t}$$

if $k \in \mathbb{N}$ and $k > \frac{n}{2m}$. Note first that

$$K(t, f) \le \max(1, t/s) K(s, f). \quad (9.3)$$

Indeed, if $t/s \ge 1$ then

$$K(t, f) = \inf (\|f_0\| + t \|f_1\|) = \inf \left(\|f_0\| + \frac{t}{s} s \|f_1\|\right) \le \frac{t}{s} K(s, f).$$

If $t/s < 1$ then immediately $K(t, f) \le K(s, f)$. Using (9.3) we obtain

$$K(2^{-k\nu}, f) \le \max\left(1, \frac{2^{-k\nu}}{t}\right) K(t, f)$$

or

$$2^{k\nu} K(2^{-k\nu}, f) \le t^{-1} K(t, f), \quad t < 2^{-k\nu}.$$

Next,

$$\int_0^\infty t^{-\frac{n}{2mk}} K(t,f)\frac{dt}{t} = \sum_{\nu=-\infty}^{\infty} \int_{2^{-k(\nu+1)}}^{2^{-k\nu}} t^{-\frac{n}{2mk}} K(t,f)\frac{dt}{t}.$$

Therefore,

$$2^{\nu\frac{n}{2m}} K(2^{-\nu k},f) = 2^{\nu k} K(2^{-\nu k},f) 2^{\nu\left(\frac{n}{2m}-k\right)} \le t^{-1} K(t,f) 2^{\nu\left(\frac{n}{2m}-k\right)} \le ct^{-1}K(t,f)t^{-\frac{n}{2mk}+1}$$

for $2^{-k(\nu+1)} < t < 2^{-k\nu}$ and $k > \frac{n}{2m}$. It implies that

$$2^{\nu\frac{n}{2m}} K(2^{-\nu k},f)\frac{1}{t} \le ct^{-\frac{n}{2mk}} K(t,f)\frac{1}{t},$$

where $2^{-k(\nu+1)} < t < 2^{-k\nu}$ and $k > \frac{n}{2m}$. Integrating this inequality with respect to t over the interval in question yields

$$2^{\nu\frac{n}{2m}} K(2^{-\nu k},f) \int_{2^{-k(\nu+1)}}^{2^{-k\nu}} \frac{dt}{t} \le c \int_{2^{-k(\nu+1)}}^{2^{-k\nu}} t^{-\frac{n}{2mk}} K(t,f)\frac{dt}{t}$$

or

$$2^{\nu\frac{n}{2m}} K(2^{-\nu k},f) \le c' \int_{2^{-k(\nu+1)}}^{2^{-k\nu}} t^{-\frac{n}{2mk}} K(t,f)\frac{dt}{t}.$$

Finally we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} |f_j e_j| &\le c_1 \sum_{\nu=-\infty}^{\infty} 2^{\nu\frac{n}{2m}} K(2^{-\nu k},f) \le c_1 \sum_{\nu=-\infty}^{\infty} \int_{2^{-k(\nu+1)}}^{2^{-k\nu}} t^{-\frac{n}{2mk}} K(t,f)\frac{dt}{t} \\ &= c_1 \int_0^\infty t^{-\frac{n}{2mk}} K(t,f)\frac{dt}{t}. \end{aligned}$$

But the latter integral is finite if and only if $f \in D^{\frac{n}{2m},1}$. □

Let $\{\varphi_j(\xi)\}_{j=0}^{\infty}$ be a *partition of unity* i.e. $\sum_{j=0}^{\infty} \varphi_j(\xi) \equiv 1, \xi \in \mathbb{R}^n, \varphi_j \ge 0, \varphi_j \in C_0^\infty(\mathbb{R}^n), \varphi_j(\xi) = \varphi(2^{-j}\xi), j = 1,2,\ldots$, where $\varphi \in C_0^\infty$ with $\operatorname{supp}\varphi \subset \{1/4 \le |\xi| \le 1\}$ and $\operatorname{supp}\varphi_0 \subset \{|\xi| < 1\}$.

Definition. Let $s \in \mathbb{R}$ and $1 \le p \le \infty$. A function f belongs to the *Besov space* $B^s_{2,p}(\mathbb{R}^n)$ for $1 \le p < \infty$ if

$$\|f\|_{B^s_{2,p}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} 2^{jps} \left(\int_{\mathbb{R}^n} |f_j(x)|^2 dx\right)^{p/2}\right)^{1/p} < \infty,$$

where $f_j(x) = F^{-1}\left(\varphi_j(\xi) Ff\right)(x)$. The space $B^s_{2,\infty}(\mathbb{R}^n)$ is defined to consist of functions f for which

$$\|f\|_{B^s_{2,\infty}(\mathbb{R}^n)} := \sup_{0\le j\le\infty} 2^{js} \left(\int_{\mathbb{R}^n} |f_j(x)|^2 dx\right)^{1/2} < \infty$$

with f_j as above.

Exercise 49. Prove that

$$B^s_{2,p}(\mathbb{R}^n) \subset B^s_{2,\infty}(\mathbb{R}^n)$$

for $1 \le p < \infty$.

In the case $1 \le p < \infty$ the previous definition is equivalent to

$$\left[\sum_{j=1}^{\infty} 2^{j\left(ps+\frac{np}{2}\right)} \left(\int_{1/4\le|\eta|\le 1} |\varphi(\eta)Ff(2^j\eta)|^2 d\eta\right)^{p/2} + \left(\int_{|\xi|<1} |\varphi_0(\xi)Ff(\xi)|^2 d\xi\right)^{p/2}\right]^{1/p} < \infty$$

while in the case $p = \infty$ we might equivalently require that

$$\left(\int_{1/4\le|\eta|\le 1} |\varphi(\eta)Ff(2^j\eta)|^2 d\eta\right)^{1/2} \le C2^{-js}$$

and

$$\left(\int_{|\xi|<1} |\varphi_0(\xi)Ff(\xi)|^2 d\xi\right)^{1/2} \le C$$

for some constant C.

Exercise 50. Prove that

$$B^s_{2,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$$

if $s > n/2$ and $1 \le p \le \infty$.

Remark. It was proved by Peetre and O.V. Besov and S.M. Nikol'skii that $f \in B^s_{2,p}(\mathbb{R}^n), s > 0, 1 \le p \le \infty$ if and only if

$$\|f\|_{B^s_{2,p}} \asymp \|f\|_{L^2} + \left(\int_0^\infty \left[\frac{\omega^{(2)}(t; D^\alpha f)}{t^{s-k}}\right]^p \frac{dt}{t}\right)^{1/p},$$

where $k \in \mathbb{N}_0, |\alpha| = k, s - k > 0$ and

$$\omega^{(2)}(t; g) := \sup_{|h|<t} \|g(\cdot + h) - 2g + g(\cdot - h)\|_{L^2}.$$

Let $\omega(\xi)$ from Example 9.2 be equal to $|\xi|^m$ with $m > 0$. Then

$$G(t) = F^{-1}(e^{-t|\xi|^m} F)$$

is the semi-group with the infinitesimal generator

$$Af = -F^{-1}(|\xi|^m Ff) = -(-\Delta)^{m/2} f.$$

It is also true that

$$D(A) = \left\{f \in L^2(\mathbb{R}^n) : |\xi|^m Ff \in L^2\right\} = W^m_2(\mathbb{R}^n).$$

For this generator A the following theorem holds.

Theorem 6. *If* $0 < \theta < 1, 1 \leq p \leq \infty$ *and* $m > 0$ *then*

$$(L^2(\mathbb{R}^n), D(A))_{\theta,p} = B^{m\theta}_{2,p}(\mathbb{R}^n).$$

Proof. Let $f \in B^{m\theta}_{2,p}(\mathbb{R}^n)$ and let $\{\varphi_j\}_{j=0}^{\infty}$ be a partition of unity. Then

$$Ff = \sum_{j=0}^{\infty} \varphi_j Ff$$

so that

$$\|f\|_{L^2} = \|Ff\|_{L^2} \leq \sum_{j=0}^{\infty} \|\varphi_j Ff\|_{L^2} \leq \sum_{j=0}^{\infty} \|f_j\|_{L^2}.$$

Moreover,

$$Af_j = AF^{-1}(\varphi_j Ff) = -F^{-1}(|\xi|^m F(F^{-1}\varphi_j Ff)) = -F^{-1}(|\xi|^m \varphi_j Ff).$$

Hence

$$\|Af_j\|_{L^2} = \||\xi|^m \varphi_j Ff\|_{L^2} \leq 2^{jm} \|f_j\|_{L^2}$$

because $\operatorname{supp} \varphi_j \subset \{2^{j-2} \leq |\xi| \leq 2^j\}, j = 1, 2, \ldots$, $\operatorname{supp} \varphi_0 \subset \{|\xi| < 1\}$ and, therefore,

$$|\xi|^m \leq 2^{jm}$$

for $\xi \in \operatorname{supp} \varphi_j$. Next we prove that

$$h(t, f) \leq c \sum_{j=0}^{\infty} \min(1, t2^{jm}) \|f_j\|_{L^2}. \tag{9.4}$$

Indeed, if $t \geq 1$ then

$$h(t, f) = \sup_{s<t} \|G(s)f - f\| \leq (M + 1) \|f\|_{L^2} \leq (M + 1) \sum_{j=0}^{\infty} \|f_j\|_{L^2}.$$

If $0 < t < 1$ then

$$\begin{aligned} h(t, f) &= \sup_{s<t} \left\| \int_0^s G(\tau) Af d\tau \right\|_{L^2} \leq M \sup_{s<t} \int_0^s \|Af\|_{L^2} d\tau \leq Mt \|Af\|_{L^2} \\ &\leq Mt \sum_{j=0}^{\infty} \|Af_j\|_{L^2} \leq Mt \sum_{j=0}^{\infty} 2^{jm} \|f_j\|_{L^2}. \end{aligned}$$

This proves (9.4). We also know from Theorem 4 that

$$\|f\|_{(L^2, D(A))_{\theta,p}} \asymp \|f\|_{L^2} + \left(\int_0^{\infty} \left(t^{-\theta} h(t, f)\right)^p \frac{dt}{t} \right)^{1/p}.$$

It follows from (9.4) that

$$\begin{aligned}\left(\int_0^\infty \left(t^{-\theta}h(t,f)\right)^p \frac{dt}{t}\right)^{1/p} &\le c\left(\int_0^\infty \left(t^{-\theta}\sum_{j=0}^\infty \min(1,t2^{jm})\,\|f_j\|_{L^2}\right)^p \frac{dt}{t}\right)^{1/p}\\ &\le c\sum_{j=0}^\infty \left(\int_0^\infty \left(t^{-\theta}\min(1,t2^{jm})\right)^p \frac{dt}{t}\right)^{1/p} \|f_j\|_{L^2}.\end{aligned}$$

Let us calculate the integral appearing in the last estimate. We have

$$\begin{aligned}\left(\int_0^\infty \left(t^{-\theta}\min(1,t2^{jm})\right)^p \frac{dt}{t}\right)^{1/p} &= \left(\int_0^{2^{-jm}} t^{(1-\theta)p-1}dt\right)^{1/p} 2^{jm}\\ &+ \left(\int_{2^{-jm}}^\infty t^{-\theta p-1}dt\right)^{1/p}\\ &= c' 2^{-(1-\theta)jm}2^{jm} + c'' 2^{j\theta m} = c2^{j\theta m}.\end{aligned}$$

Thus

$$\|f\|_{(L^2(\mathbb{R}^n),D(A))_{\theta,p}} \le \|f\|_{L^2} + c\sum_{j=0}^\infty 2^{jm\theta}\,\|f_j\|_{L^2}.$$

Hence

$$\|f\|_{(L^2(\mathbb{R}^n),D(A))_{\theta,p}} \le c\,\|f\|_{B^{m\theta}_{2,1}(\mathbb{R}^n)}.$$

This inequality means that

$$B^{m\theta}_{2,1}(\mathbb{R}^n) \subset (L^2(\mathbb{R}^n),D(A))_{\theta,p}.$$

It is also possible to prove sharper embedding

$$B^{m\theta}_{2,1}(\mathbb{R}^n) \subset B^{m\theta}_{2,p}(\mathbb{R}^n) \subset (L^2(\mathbb{R}^n),D(A))_{\theta,p}.$$

It remains to prove the opposite inequality, namely

$$\|f\|_{B^{m\theta}_{2,p}(\mathbb{R}^n)} \le c\,\|f\|_{(L^2(\mathbb{R}^n),D(A))_{\theta,p}}.$$

To that end, note that

$$\begin{aligned}f_j(x) &= F^{-1}(\varphi_j Ff) = F^{-1}(\varphi_j(e^{-t|\xi|^m}-1)^{-1}F(G(t)f-f))\\ &\overset{t=2^{-jm}}{=} F^{-1}(\varphi_j(e^{-(|\xi|2^{-j})^m}-1)^{-1}F(G(2^{-jm})f-f)).\end{aligned}$$

Since $\operatorname{supp}\varphi_j \subset \{2^{j-2} \le |\xi| \le 2^j\}$ then

$$\frac{1}{4} \le |\xi|2^{-j} \le 1$$

or

$$\left(\frac{1}{4}\right)^m \le (|\xi|2^{-j})^m \le 1$$

for $\xi \in \operatorname{supp} \varphi_j$. Hence

$$\begin{aligned} \|f_j\|_{L^2} &= \left\| \varphi_j (e^{-(|\xi| 2^{-j})^m} - 1)^{-1} F(G(2^{-jm})f - f) \right\|_{L^2} \\ &\le \frac{1}{1 - e^{-(1/4)^m}} \left\| F(G(2^{-jm})f - f) \right\|_{L^2} \le c \left\| G(2^{-jm})f - f \right\|_{L^2} \le c h(2^{-jm}, f). \end{aligned}$$

By the definition of Besov spaces we have

$$\|f\|_{B^s_{2,p}} = \left(\sum_{j=0}^{\infty} 2^{jps} \|f_j\|^p_{L^2} \right)^{1/p} \le c \left(\sum_{j=0}^{\infty} 2^{jps} h(2^{-jm}, f)^p \right)^{1/p}.$$

As in the proof of Theorem 5 it can be checked that the convergence of the latter series is equivalent to the convergence of the integral

$$\left(\int_0^\infty \left(t^{-\theta} h(t, f) \right)^p \frac{dt}{t} \right)^{1/p}$$

with $s = m\theta, 0 < \theta < 1$.

Exercise 51. Prove this fact.

Thus, the required inequality is proved and, therefore, we have the embedding

$$(L^2(\mathbb{R}^n), D(A))_{\theta,p} \subset B^{m\theta}_{2,p}(\mathbb{R}^n).$$

This finishes the proof. □

Corollary. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary then*

$$(L^2(\Omega), \overset{\circ}{W}{}^m_2(\Omega))_{\theta,p} = \overset{\circ}{B}{}^{m\theta}_{2,p}(\Omega)$$

for any $m > 0$ such that $m\theta - 1/2$ is not an integer.

Let A be a self-adjoint and non-negative extension of an elliptic differential operator of even order m. Since

$$\overset{\circ}{W}{}^{mk}_2(\Omega) \subset D(A^k)$$

for any $k \in \mathbb{N}$ then

$$\left(L^2(\Omega), \overset{\circ}{W}{}^{mk}_2(\Omega) \right)_{\theta,p} \subset \left(L^2(\Omega), D(A^k) \right)_{\theta,p}$$

or

$$\overset{\circ}{B}{}^{mk\theta}_{2,p}(\Omega) \subset D^{\theta k, p}.$$

Peetre's theorem means that the corresponding Fourier series converges absolutely and uniformly if $f \in D^{\frac{n}{2m},1}$. This implies that the same is true for $f \in \overset{\circ}{B}{}^{n/2}_{2,1}(\Omega)$. We proved also that if $f \in D(A^\sigma)$ with $\sigma > \frac{n}{2m}$ then the Fourier series converges absolutely. Since

$$\left(L^2(\Omega), \overset{\circ}{W}{}^{mk}_2(\Omega) \right)_{\theta,2} \subset \left(L^2(\Omega), D(A^k) \right)_{\theta,2} = D(A^{k\theta})$$

or

$$\overset{\circ}{B}{}^{mk\theta}_{2,2}(\Omega) \subset D(A^{k\theta})$$

then the Fourier series converges absolutely for $f \in \overset{\circ}{B}{}^{n/2+\varepsilon}_{2,2}(\Omega), \varepsilon > 0$.

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