1. **The trajectory of the center of mass**
   A grenade that's falling vertically blows up in two pieces of equal size at the altitude of 2000 meters. Its velocity is 60 m/s at that time. After the explosion the other part moves at the velocity of 80 m/s downwards. Where is the center of mass of the system 10 seconds after the explosion?

2. **Angular momentum using the center of mass**
   We consider a group of particles with locations $\boldsymbol{r}_i$ and masses m_i. The total mass is M, the location of the center of mass $\boldsymbol{R}$ and the locations of particles relative to the center of mass $\boldsymbol{r}'_i$.
   a) Express $\boldsymbol{r}_i$ using $\boldsymbol{R}$ and $\boldsymbol{r}'_i$.
   b) Express the velocity $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i$ of a particle i using the velocity of the center of mass $\boldsymbol{V} = \dot{\boldsymbol{R}}$ and the velocity of the particle with respect to the center of mass $\boldsymbol{v}'_i = \dot{\boldsymbol{r}}'_i$. Here the time derivative is denoted by a dot.
   c) Show that $\sum_i m_i \boldsymbol{r}'_i = 0$ and $\sum_i m_i \boldsymbol{v}'_i = 0$.
   d) Show that the total angular momentum $\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i$ can be written using the center of mass as

   $$\boldsymbol{L} = \boldsymbol{R} \times M\boldsymbol{V} + \sum_i \boldsymbol{r}'_i \times \boldsymbol{p}'_i \ ,$$

   where $\boldsymbol{p}'_i = m_i \boldsymbol{v}'_i$ is the momentum of particle i with respect to the center of mass.

3. **Kinetic energy using the center of mass**
   Show that the kinetic energy of a many body system can be written as

   $$T = \frac{1}{2}MV^2 + \frac{1}{2}\sum_i m_i v_i'^2 \ ,$$

   where M is the total mass of the system, $\boldsymbol{V}$ is the velocity of the center of mass and $\boldsymbol{v}'_i$ is the velocity of the ith particle relative to the center of mass of the system. Notice that $V^2 = \boldsymbol{V} \cdot \boldsymbol{V}$ and $v_i'^2 = \boldsymbol{v}'_i \cdot \boldsymbol{v}'_i$. Hint: use the results of the previous problem.

4. **Force field**

   a) Is the following force field $\boldsymbol{F}$ conservative? Find the potential associated with the $\boldsymbol{F}$.

   $$\boldsymbol{F} = (6abz^3y - 20bx^3y^2)\boldsymbol{i} + (6abxz^3 - 10bx^4y)\boldsymbol{j} + (18abxz^2y)\boldsymbol{k}$$

b) A particle of mass m lies at the origin of the coordinate system. It's potential of the gravitational field is given by

$$V(r) = -\gamma \frac{m}{r}, \quad r^2 = x^2 + y^2 + z^2.$$

Form the components of the force field vector $\boldsymbol{F}$ and show that $\boldsymbol{F}$ is conservative.

c) Let a particle be in a force field

$$\boldsymbol{F} = 24t^2\boldsymbol{i} + (36t - 16)\boldsymbol{j} - 12t\boldsymbol{k}$$

and its velocity is

$$\boldsymbol{v} = 4t^3\boldsymbol{i} + (9t^2 - 8t)\boldsymbol{j} - 3t^2\boldsymbol{k}.$$

What is the work done by the force field when moving the particle from the point $t = 1$ to the point where $t = 2$?
Hint: Rewrite the equation of work using time t as integration variable.

5. **Plane polar coordinates**
Let (r, θ) represent the polar coordinates describing the position of a particle. If $\hat{\boldsymbol{r}}$ is a unit vector in the direction of the position vector $\boldsymbol{r}$ and $\hat{\boldsymbol{\theta}}$ is a unit vector perpendicular to $\boldsymbol{r}$ and in the direction of increasing angle θ, show that

$$\begin{aligned} \hat{\boldsymbol{r}} &= \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta \\ \hat{\boldsymbol{\theta}} &= -\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta \end{aligned}$$

$$\begin{aligned} \boldsymbol{i} &= \hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \\ \boldsymbol{j} &= \hat{\boldsymbol{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta. \end{aligned}$$

In addition, prove relations

$$\begin{aligned} \dot{\hat{\boldsymbol{r}}} &= \hat{\boldsymbol{\theta}}\dot{\theta} \\ \dot{\hat{\boldsymbol{\theta}}} &= -\hat{\boldsymbol{r}}\dot{\theta} \end{aligned}$$

and show that (in polar coordinates) the velocity and the acceleration are given by

$$\begin{aligned} \boldsymbol{v} &= \hat{\boldsymbol{r}}\dot{r} + \hat{\boldsymbol{\theta}}r\dot{\theta} \\ \boldsymbol{a} &= \hat{\boldsymbol{r}}(\ddot{r} - r\dot{\theta}^2) + \hat{\boldsymbol{\theta}}(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \end{aligned}$$

1. **Snell's law**
A particle with total energy E lies in a potential

$$V(x,y,z) = \begin{cases} V', & \text{if } z < 0 \\ V, & \text{if } z > 0 \end{cases}.$$

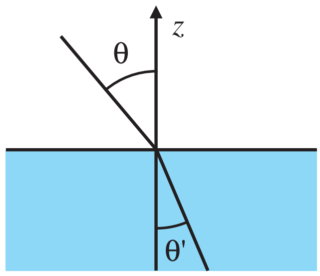
Show that the particle's change of the direction of the motion at the boundary $z = 0$ obeys the Snell's law

$$n \sin\theta = n' \sin\theta',$$

where

$$\frac{n'}{n} = \sqrt{\frac{E - V'}{E - V}}.$$

Hint: Use the conservation of the momentum components p_x and p_y and the conservation of energy.



2. **Angular momentum in moving frame**
Let's examine a coordinate system moving (but not rotating) with point $\boldsymbol{r}_0$. Show that the angular momentum $\boldsymbol{L}' = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_0) \times (\boldsymbol{p}_i - m_i \dot{\boldsymbol{r}}_0)$ in this coordinate system can be expressed as

$$\dot{\boldsymbol{L}}' = \boldsymbol{N}' + M(\boldsymbol{r}_0 - \boldsymbol{R}) \times \ddot{\boldsymbol{r}}_0,$$

where $\boldsymbol{N}'$ is the torque of the exterior forces relative to point $\boldsymbol{r}_0$, M is the total mass and $\boldsymbol{R}$ is the center of gravity. Specify two special cases where $\dot{\boldsymbol{L}}' = \boldsymbol{N}'$ is valid?

3. **Perturbation calculation of a trajectory**
Calculate through the intermediate steps in the example problem of perturbation calculation presented in Section 2.3 of the lecture notes. Hint: the time spent during the throw is determined by the condition $y_0(t) + y_1(t) = 0$. This leads to second order equation for t. It is possible to solve this by the standard formula. It is, however, much simpler to apply the perturbation method also to this equation, i.e. you substitute $t = t_0 + t_1$ and form the zeroth and first order equations in k.

4. **Perturbation calculation of a nonlinear oscillator**
Let's investigate one dimensional motion of the particle (mass m) in a potential of the form

$$U(x) = \frac{1}{2}kx^2 - \frac{1}{4}m\varepsilon x^4.$$

In addition, the particle is affected by the harmonically oscillating force $mA\cos\omega t$.

a) Write the particle's Newton's equation of the motion.

b) Let us assume that $\omega_0^2 = k/m \neq \omega^2$. Write the zeroth and the first order equations of the motion with the help of perturbation theory (with respect to ε) and solve them with the following ansatz

$$x_0 = B\cos\omega t, \quad x_1 = C\cos\omega t + D\cos 3\omega t.$$

Sketch the amplitudes B, C and D of the vibration as a function of frequency (in the case $\varepsilon > 0$). For which values of ω the first order perturbation theory isn't sufficient?

1. **Plane polar coordinates**
   Calculate through the intermediate steps in the example 2 of section 3.2: *One particle in polar coordinates.*

2. **Trajectory in Lagrangian mechanics**

   a) Use Lagrange's equations to describe the motion of a projectile launched with speed v_0 at angle α with the horizontal.

   b) Try to find some term which would produce the effect of the air resistance if added to the Lagrange's function.

3. **Pendulum**
   Set up the Lagrangian for a simple pendulum and obtain an equation describing it's motion. Solve the equation of the motion in the case of small vibrations. (Amplitude of the vibration $\ll$ length of the wire)

4. **Spring**

   a) Show that the potential energy of a spring with spring constant k is $\frac{1}{2}ks^2$, where s is the change of length of the spring form its equilibrium length. For that calculate the work done by an external force when the spring is stretched by s, and verify that the same calculation work for compression as well.

   b) Write Lagrange's function for a weight suspended to a spring so that only vertical motion has to be considered. Solve Lagrange's equations.
   Hint: A differential equation of the form $Ly = f$, where L is a linear operator and f is an y-independent "inhomogeneity", can be solved by first finding the general solution of the homogeneous equation $Ly = 0$ and then one solution for the full equation $Ly = f$. The general solution is the sum of the two solutions.

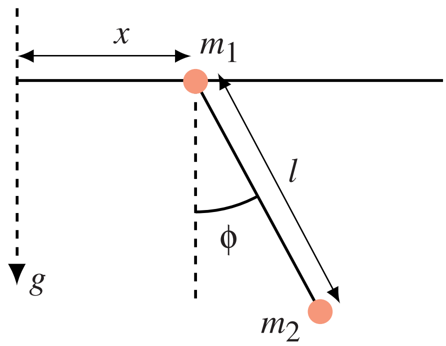
5. **Spherical coordinates**

   a) Show that the kinetic energy of a particle in a spherical coordinate system ($\boldsymbol{r} = r\cos\varphi\sin\theta\ \hat{\boldsymbol{x}} + r\sin\varphi\sin\theta\ \hat{\boldsymbol{y}} + r\cos\theta\ \hat{\boldsymbol{z}}$) is

   $$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2).$$

   b) Write a Lagrange's function for a particle which is in homogeneous gravity field and is connected to a fixed point by a massless spring (like the one in Problem 4).

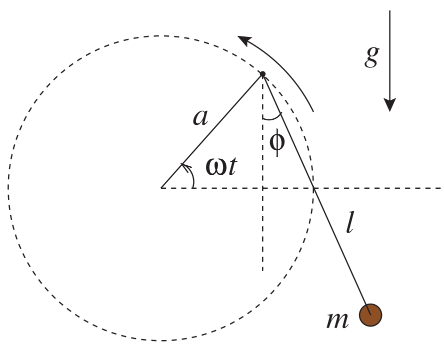
1. **Gliding pendulum**
   Consider a system that consists of a simple pendulum (mass m_2, length l) and the point particle (mass m_1) to which the pendulum is attached. This latter particle can move horizontally in the plane of pendulum motion (figure).



   Show that the corresponding Lagrangian is:

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2 \dot{\phi}^2 + m_2 l \dot{x}\dot{\phi}\cos\phi + g m_2 l \cos\phi.$$

2. **Rotating pendulum**
   Consider a plane pendulum (mass m, length l), whose suspension point moves with constant angular speed ω along a circle (radius a) in the plane of pendulum.



   Show that the Lagrangian is:

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mal\omega\dot{\phi}\sin(\phi - \omega t) + mgl\cos\phi - mga\sin\omega t + \frac{1}{2}ma^2\omega^2.$$

   Why don't the two last terms have any effect on the Lagrange's equations of motion?

TURN

3. **Electromagnetic potentials**

   a) Show that the Maxwell equations

$$\nabla \cdot \boldsymbol{B} = 0 \quad , \qquad \nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}$$

   are automatically fulfilled, when the fields are written using a scalar potential ϕ and a vector potential $\boldsymbol{A}$ as

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial}{\partial t}\boldsymbol{A} \quad , \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

   b) Show that $\boldsymbol{A}$ ja ϕ are not unambigiously defined, but that the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ do not change in a *gauge transformation*, where

$$\boldsymbol{A'} = \boldsymbol{A} + \nabla\chi \quad , \qquad \phi' = \phi - \frac{\partial\chi}{\partial t}.$$

   Here $\chi(\boldsymbol{r}, t)$ is an arbitrary function.

4. **Motion in magnetic field**
   Examine the motion of a charged particle in a homogenous magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. Show that $\boldsymbol{B}$ can be expressed by the vector potential $\boldsymbol{A} = -By\hat{\boldsymbol{x}}$. Write the Lagrange's equations in cartesian coordinates. Show that the solution in the x-y plane is a circular motion

$$x = x_0 + r_0\cos(\omega t + \phi_0) \quad , \quad y = y_0 + r_0\sin(\omega t + \phi_0)$$

   at the angular speed

$$\omega = -\frac{qB}{m},$$

   which is independent of the radius r_0 of the circular motion. What can you say about the motion in z -direction? Verify that the general solution has the correct amount of free parameters (6).

1. **Harmonic motion**
   Form the Lagrangian, calculate the equations of motion and express the general solution in the following cases
   a) particle in a one-dimensional potential
   $$V(x) = 6kx(x-2),$$
   where k is a constant.
   b) three-dimensional harmonic potential
   $$V(x,y,z) = \frac{1}{2}(k_1x^2 + k_2y^2 + k_3z^2),$$
   where k_1, k_2 and k_3 are constants.

2. **Motion on a cylinder**
   A particle of mass m moves on the outer (frictionless) surface of the cylinder (cylinder's axis is in z-direction) $x^2 + y^2 = R^2$ under the influence of a force $\boldsymbol{F} = -k\boldsymbol{r}$ which is directed towards the origin. Find the Lagrangian of the system. Furthermore, obtain the equations of motion and solve them. Use cylindrical polar coordinates $\boldsymbol{r} = \hat{\boldsymbol{x}}\rho\cos\varphi + \hat{\boldsymbol{y}}\rho\sin\varphi + \hat{\boldsymbol{z}}z$.

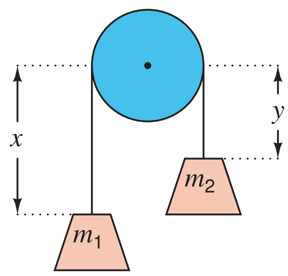
3. **Minimal surface of revolution**
   Calculate through the intermediate steps of the variation calculation of example 3 in Section 4.2 in the lecture notes. In showing the general solution of the differential equation it is sufficient to show that the solution satisfies the differential equation and contains two constants (whose effect is not the same). [This part has been translated to English.]

4. **Fermat's law**
   Fermat's law says that light travels between two points along a route, so that the time used is extremal (minimum). Starting from Fermat's law
   a) prove, that in homogenous medium (light speed c) light travels along a straight line. (Hint: reduce this problem to a known example that you need not solve again.)
   b) derive the law of reflection and Snell's refraction law in a boundary surface of two mediums (light's speeds c_1 ja c_2).

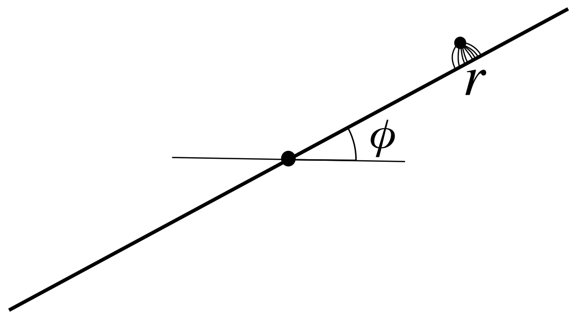
1. **Second derivatives in Lagrangian**
   Let's assume that the Lagrange's function depends on accelerations i.e. $L = L(q, \dot{q}, \ddot{q}; t)$. Derive the equation of motion for the function q starting from Hamilton's principle. What boundary conditions are reasonable in this case? Furthermore, what is the equation of motion in the case
   $$L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2 \ ?$$

2. **Pendulum using differential constraints**
   Examine the simple pendulum again so that the length of the pendulum is taken as a differential constraint in Lagrange's equations. Prove that the equation of motion for the pendulum is the same as obtained before and calculate the generalized force streching the string of the pendulum. [Answer: $Q = mr\dot{\theta}^2 + mg\cos\theta$.]

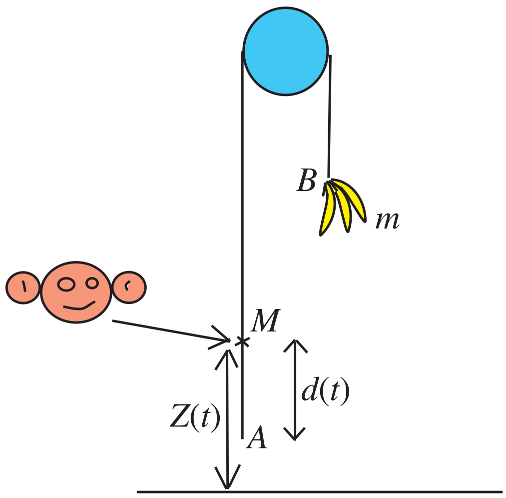
3. **Atwood's machine using differential constraints**
   Let's consider again Atwood's machine investigated in lectures. In order to calculate the tension Q in the string write the generalized coordinates for the locations of both masses. Take into account the fixed length of the string with differential constraints. [Answer: $Q = 2m_1m_2g/(m_1 + m_2)$.]



4. **Walking spider**
   A spider walks along a stem of grass. The stem can rotate around a horizontal axis attached to its center of mass. The spider and the stem are in uniform gravity. What has to be the spider's distance r from the rotation axis so that the angular velocity $\dot{\phi}$ of the stem would be constant? (Hint: the walk of the spider along the stem defines a function $r(t)$, so the only free generalized coordinate is the stem's angle ϕ. Solve the corresponding Lagrange's equation.) Prove also, that the stem's kinetic energy caused by the rotation $T_{\text{stem}} = \frac{1}{2}I\dot{\phi}^2$ doesn't change the final result. [Answer: $r = r_0 - (g/2\omega^2)\sin\phi$.]


ϕ
r

1. **Monkey and the bananas**
A massless inextensible string passes over a pulley which is a fixed distance above the floor. A bunch of bananas of mass m is attached to one end B of the string. A monkey of mass M is initially at the other end A. The monkey climbs the string, and his displacement $d(t)$ with respect to the end A is a given function of time.The system is initially at rest, so that the initial conditions are $d(0) = \dot{d}(0) = 0$. Introduce suitable generalized coordinates and calculate the lagrangian of the system in terms of these coordinates. Show that the equation of motion governing the height Z of the monkey above the floor is
$$(m + M)\ddot{Z} - m\ddot{d} = (m - M)g.$$
Integrate the equation to find the subsequent motion. In the special case that $m = M$, show that the bananas and the monkey rise through equal distances so that the vertical separation between them is constant.



2. **Constants of motion**
In exercise 4.1 we derived the Lagrangian
$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 l\dot{x}\dot{\phi}\cos\phi + gm_2 l\cos\phi.$$
What are the constants of motion?

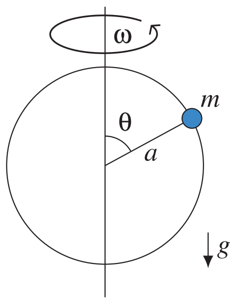
3. **Relativistic particle**
We claim that a relativistic particle in a potential $V = -k/r$ is described by Lagrangian
$$L(r, \phi, \dot{r}, \dot{\phi}) = -mc^2\sqrt{1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}} + \frac{k}{r}, \tag{1}$$

where r and ϕ are the polar coordinates, m the mass of the particle and c the velocity of light. Find the constants of motion. Are these in agreement with what you learned in Introduction to relativity I course? Note that for a relativistic particle, H has to be calculated starting from the definition of H.

4. **Hamiltonian for a time dependent consraint**
We study a pearl (mass m) that can slide without friction on a vertical circle. The radius of the circle is a and it rotates around the vertical axis (passing trough its center) with angular velocity ω. Using the expression for kinetic energy in spherical coordinates, show that the system is described by the Lagrangian

$$L = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\omega^2 \sin^2\theta - mga\cos\theta. \tag{2}$$

Form expressions for the Hamiltonian and the total energy of the pearl. Show that they are not the same. Which of them is constant?



1. **The ratio of the masses of the central bodies from satellite orbits**
   Calculate an approximation for the ratio of Earth's and Sun's masses by using only the length of a year, month (27.3 days), Earth's orbit's approximate radius ($149 \cdot 10^6$ km) and the radius of Moon's orbit (380000 km).

2. **Satellite orbit**
   From the surface of a planet with no atmosphere (mass M, radius R) a satellite with mass m ($m \ll M$) is launched into orbit. Satellite is first risen to an altitude $h = r - R$ and its then given a starting velocity v which is perpendicular to radius vector. Show that the eccentricity ϵ of the orbit as a function of starting velocity v is

   $$\varepsilon = \left| 1 - \frac{rv^2}{GM} \right| .$$

   At what values of v the orbit is an ellipse, a circle, a parabola or a hyperbola?

3. **Closed vs. unclosed orbits studied by perturbation theory**
   Calculate through the intermediate steps of section "Closing of the orbits" in Section 5.3 of the lecture notes [translated to English].

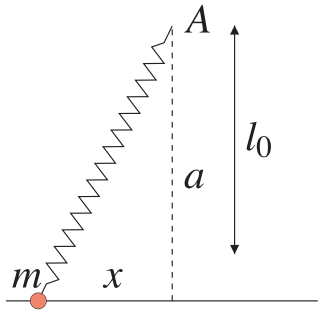
4. **Scattering from a hard sphere**
   Examine a situation where point-like particles scatter from a fixed ball, which radius is R. The scattering is assumed elastic, which means that the incident and leaving angles of a particle with respect to the surface normal are equal. Calculate differential and total cross sections of the scattering. (*Answer:* $\frac{d\sigma}{d\Omega} = \frac{R^2}{4}, \sigma = \pi R^2$.)

5. **Effect of attraction on the collision cross section**
   Show that the cross section $\sigma_{\text{collision}}$ for the case, where a point-like particle (mass m, relative velocity far away v_∞) collides a big ball (radius R, mass $M \gg m$), is given by

   $$\sigma_{\text{collision}} = \pi R^2 + \frac{2\pi GMR}{v_\infty^2} ,$$

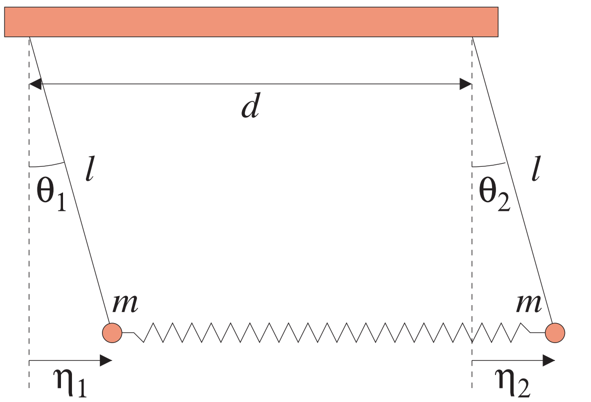
   where G is the gravitational constant. Hint: the collision takes place if the distance $f - a$ of nearest point in the particle's orbit to center of the big ball is smaller than R.

1. **Constrained oscillator**
   Find the frequency in the case of small oscillations of a particle (mass m) which is free to move along a line and is attached to a spring whose other end is fixed at a point A at distance a from the line. A force F is required to extend the spring to length a ($a > l_0$).



2. **Coupled pendula**
   Two identical pendula (length l, mass m) are coupled using a horizontal spring (spring constant k). In equilibrium the pendula are vertical. Write Lagrangian, both exactly and in the approximation of small oscillations. Lagrangian for small oscillations is:

$$L = \frac{1}{2}m(\dot{\eta_1}^2 + \dot{\eta_2}^2) - \frac{1}{2}(\frac{mg}{l} + k)(\eta_1^2 + \eta_2^2) + k\eta_1\eta_2$$

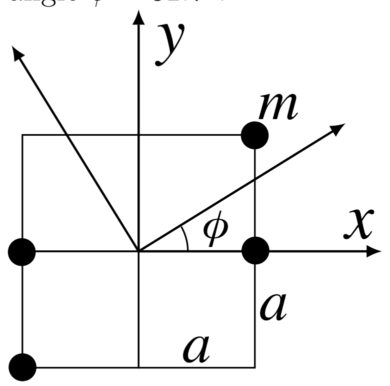


3. **Frequencies of coupled pendula**
   Show that the frequencies of small oscillations in the previous problem are $\omega_1 = \sqrt{\frac{g}{l}}$ and $\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$. What are the eigenvectors?

1. **Gravitation in a rotating frame**
Let's examine what kind of apparent forces arise because of Earth's rotation (angular velocity ω). We define $m\boldsymbol{g}$ as the force acting on a body that is stationary in the frame rotating with the earth.
   a) Show that $\boldsymbol{g} = \boldsymbol{g}_0 + r\omega^2 \sin\theta \hat{\boldsymbol{\rho}}$, where both spherical (r, θ, ϕ) and cylindrical (ρ, φ, z) coordinates have been used, and $\boldsymbol{g}_0$ is the bare gravitational acceleration.
   b) Let us form a coordinate system where $\hat{\boldsymbol{z}}$ is in the direction of $-\boldsymbol{g}$, $\hat{\boldsymbol{x}}$ to the south and $\hat{\boldsymbol{y}}$ to the east. In this coordinate system $\boldsymbol{\omega} = \omega(\hat{\boldsymbol{z}}\cos\theta - \hat{\boldsymbol{x}}\sin\theta) + O(\omega^3)$. By means of perturbation calculus, show that the trajectory of a free fall from height h is
$$\boldsymbol{r}(t) = (h - \frac{1}{2}gt^2)\hat{\boldsymbol{z}} + \frac{1}{3}\omega g t^3 \sin\theta \hat{\boldsymbol{y}}.$$

2. **Moment of inertia of a rectangular cuboid**
A rectangular cuboid is described by the density
$$\rho(\boldsymbol{r}) = \begin{cases} M/abc & \text{if } |x| < \frac{a}{2} \text{ and } |y| < \frac{b}{2} \text{ and } |z| < \frac{c}{2}, \\ 0 & \text{elsewhere,} \end{cases} \tag{3}$$
where M is the total mass. Calculate the tensor of the moment of inertia for this body with respect to its center. As a special case ($b \to 0$ ja $c \to 0$) derive the tensor of the moment of inertia for a thin rod.

3. **Principal moment of inertia for an asymmetric body**
Consider four point masses (each mass m) that are rigidly bound to each other at locations $(1, 0, 0)$, $(1, 1, 0)$, $(-1, 0, 0)$ and $(-1, -1, 0)$ (in units of a). Calculate the moment of inertia tensor. Show that the principal moments of inertia are
$$I_{1,2} = (3 \pm \sqrt{5})ma^2, \quad I_3 = 6ma^2. \tag{4}$$
Show that the smallest moment of inertia corresponds to the principal axis with polar angle $\phi = 31.7°$.

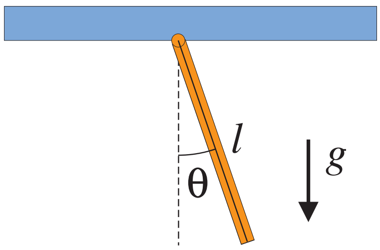


4. **The dependence of the moment of inertia on the origin**
   In an earlier exercise (1.3) we verified that the kinetic energy of a group of particles equals

$$T = \frac{1}{2} M \dot{R}^2 + T' \tag{5}$$

   where M is the total mass, $\dot{\boldsymbol{R}}$ the velocity of the center of mass and T' the kinetic energy measured with respect to the center of mass. By applying this result to a rigid body, deduce the principal moments of inertia of a thin rod with respect to its end.

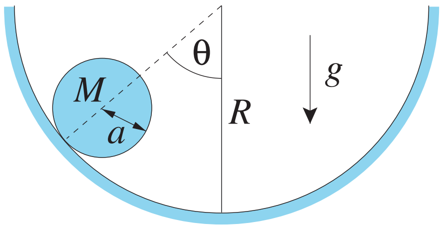
5. **Rod pendulum**
   Set up the Lagrangian for a thin uniform rod (mass M, length l) whose one end is fixed so that the rod can oscillate in a plane under in a constant gravitational field. Calculate also the frequency of small oscillation.



1. **Cylinder on a cylindrical surface**
   Consider a homogeneous cylinder (radius a, mass M) rolling on the inner surface of another cylinder (radius R). Show that its kinetic energy is

$$\frac{3}{4}M(R-a)^2\dot{\theta}^2.$$

   Show that the angular frequency of small oscillations is $\omega = \sqrt{\frac{2g}{3(R-a)}}$.



   As help you can use the result that the moment of inertia of a homogeneous cylinder around its axis is $Ma^2/2$.

2. **Euler's equation from Euler angles**
   Starting from Lagrange's equation for Euler angle γ

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\gamma}} - \frac{\partial T}{\partial \gamma} = N_3,$$

   derive the Euler equation

$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2) + N_3.$$

3. **Symmetric top in gravitational field**
   Calculate through all the phases of the example from lectures (in section 7.4) concerning a symmetric top in a gravitational field.

4. **Pendulum in Hamiltonian formalism**
   Let's examine a simple pendulum for which

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta.$$

   Write the Hamiltonian as a function of correct coordinates. Show that the Hamilton's equation of motion for θ is the same as in lagrangian formalism.

1. **Poisson bracket of a product**
Show that the Poisson bracket obeys the rule

$$[AB, C]_{\mathrm{PB}} = A\,[B, C]_{\mathrm{PB}} + [A, C]_{\mathrm{PB}}\,B.$$

2. **Conservation laws using Poisson brackets**
Show that it follows from the equation

$$\frac{dA}{dt} = -[H, A]_{\mathrm{PB}} + \frac{\partial A}{\partial t}$$

the conservation laws of (a) the Hamiltonian when $\frac{\partial H}{\partial t} = 0$ and (b) the canonical momentum corresponding to cyclic coordinate, i.e. $\frac{\partial H}{\partial q_k} = 0$.

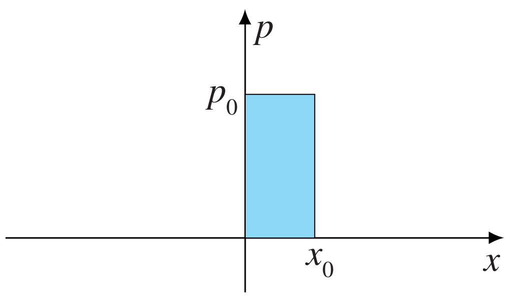
3. **Poisson brackets of angular momentum**
The angular momentum of one particle is $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$. Show that the following Poisson brackets are true for the components of angular momentum in cartesian coordinates.

$$[L_i, L_j]_{\mathrm{PB}} = \sum_k \epsilon_{ijk} L_k,$$

$$[L^2, L_i]_{\mathrm{PB}} = 0.$$

Explain why these results are valid also for a set of particles.

4. **Motion in phase space**
Consider a particle moving freely ($V = 0$) in one dimension x. Write down its Lagrangian, Hamiltonian and Hamilton's equations of motion. Draw curves $H(x, p_x) =$ constant in phase space. Suppose that the state of the system at the initial moment $t = 0$ is described by a probability distribution that is non-zero (constant) only in the rectangular area depicted below. Describe time-evolution of that region. Show that the motion satisfies Liouville's theorem. (Hint: solve equations of motion for the corners of the distribution.)



5. **Continuity equation in one dimension**
In lectures, it was mentioned that the derivation of continuity equation doesn't essentially depend on number of dimensions. Derive the continuity equation in 1-dimensional case

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{d}{dx}[\rho(x,t)v(x,t)] = 0.$$

Also express the alternative form of continuity equation.