Balmer series of the hydrogen atom

In quantum mechanics, the following second order differential equation

$$y''(r) + \left(\frac{2}{r} + E\right) y(r) = 0 \tag{1}$$

can be used to describe a hydrogen atom. This equation, the so called Schrödinger equation, is an eigenvalue problem. Its eigenvalues E give the possible energies that an electron bound to the atom can have. The eigenfunctions $y(r)$ give the wavefunctions $\psi(r) = Cy(r)/r$. The square of the wavefunction, $|\psi|^2$, gives the probabability density, which tells the probability of finding the electron in an infinitesimal volume $\mathrm{d}V$. The constant C is determined by the normalization: C is set so that the probability of finding the electron in the whole space is one,

$$4\pi \int_0^\infty |\psi(r)|^2 \, r^2 \, \mathrm{d}r = 1. \tag{2}$$

That is,

$$C = 1 \Bigg/ \sqrt{4\pi \int_0^\infty |y(r)|^2 \, \mathrm{d}r} \, . \tag{3}$$

In these equations, energy is measured in units of Rydberg, $\mathrm{Ry} = 13.606$ eV, and distance in units of Bohr radius, $a_0 = 0.52918$ Å.

Experimental information concerning the structure of the hydrogen atom can be obtained for example by measuring the emission spectra, that is, the wavelengths of the photons that the atom emits. The atom can only emit a photon if the electron makes a transition from a higher energy level E_a to a lower one E_b, $E_a > E_b$. The energy difference of the two levels, $E_a - E_b$, is transferred to the photon. When the energy of the photon E_γ is known, its wavelength λ can be found: $\lambda = ch/E_\gamma$. The constant c is the speed of light, $c = 2.9979 \cdot 10^{18}$ Å/s and h is Planck's constant, $h = 4.1357 \cdot 10^{-15}$ eVs. Therefore, using two eigenvalues of the Schrödinger equation (1), we get for the wavelength

$$\lambda = \frac{911.2 \text{ Å}}{E_a - E_b}. \tag{4}$$

Emission lines of the Balmer series are formed as the electron makes a transition from some higher state to the first excited state (that is, to the second energy level). Your task is to find the two longest wavelengths of the Balmer series. For that, you need to find the energies of the second lowest level and the following two levels (if ground state is E_0, you need to find E_1, E_2, and E_3). As your results, give the energies of the aforementioned states and the wavefunctions, normalized according to Eq. (2). Also give the two longest wavelengths of the Balmer series.

Hint: You can use the iteration method given in the lecture notes to find the eigenstates. In order to verify your results you can use the analytical solution for the enrgy levels: $E_n = -1/(n+1)^2$, where $n = 0, 1, 2, \ldots$ and the unit is Rydberg.