Excercise 5

Short Intro

Derivatives

Definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

We can find approximate values for dervivatives of some function by using Taylor-series:

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i$$

- 1st derivative
$$f'(x_i) = \frac{f_{i+1} - f_{i-1}}{2h} - \frac{1}{6}f'''(x_i)h^2 + \dots \tag{1}$$
- 2nd derivative
$$f''(x_i) = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} - \frac{1}{12}f^{(4)}(x_i)h^2 - \dots \tag{2}$$
- 3rd derivative
$$f'''(x_i) = \frac{f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2}}{2h^2} - \frac{1}{4}f^{(5)}(x_i)h^2 + \dots \tag{3}$$
- If we want calculate derivatives at the end points of the data $f'(x_1)$ and $f'(x_n)$, we must set some boundary conditions eg. $f(x_1 - h) = f(x_n + h) = 0$.
- Note: You can also define interpolating function and derivate it.

Integration

- Definite Integral of some fuctions is defined as limit of Riemann sum
$$\int_a^b f(x)dx = \sum_{i=1}^{n-1} f(t_i)(x_{i+1} - x_i)$$
where step $[x_i, x_{i+1}]$ gets smaller. Boundaries: $x_1 = a$, $x_n = b$

- Let $x_{i+1} - x_i = h$, $i = 1, \ldots, n-1$ and $f(x_i) \equiv f_i$. By setting $t_i = x_i$, we get simple approximation

$$\int_{x_1}^{x_n} f(x)\,dx \approx h \sum_{i=1}^{n-1} f_i$$

- Better approximations could be made by interpolating data with piecewise defined function.
  - If q_i are 1st order polynomia, $q_i(x) = a_i(x - x_i) + b_i$, we get so called *Trapezoidal rule*:

$$\int_{x_1}^{x_n} f(x)\,dx \approx h \sum_{i=1}^{n-1} \frac{f_i + f_{i+1}}{2} = \frac{f_1}{2} + \sum_{i=2}^{n-1} f_i + \frac{f_n}{2}$$

ex.

1. Read Sections 1.5.2-3 ja 1.6.2. and familiarise yourself with functions `D`, `Dt`, `Integrate` ja `NIntegrate`.
2. Let $f(x) = \sin\frac{1}{x}$. Make (again) a table of data points $(x_i, f(x_i))$ when $a \le x_i \le 2$, $i = 1, \ldots, n$, $a = 0.5 \ldots 0.01$ and $n = 10 \ldots 50$. (You can choose values freely).

   Calculate 1st order approximations of $f'(x_i)$ and $f''(x_i)$ with formulas derived from Taylor-series. Listplot original function and derivatives. Compare approximation with derivative calculated with build-in function.
3. Radioactive decay obeys differential equation:

$$\frac{dN(t)}{dt} = -\lambda N(t),$$

   where $N(t)$ is number of active nuclei at time t. λ is decay constant: $\lambda = (\ln 2)/T_{1/2}$ and $T_{1/2}$ is half-life. Let $\lambda = 0.1\,\mathrm{s}^{-1}$ and $N(0) = 10^{25}$.

   Replace derivative dN/dt with approximation

$$N'(t_i) = \frac{N_{i+1} - N_i}{h}$$

   and solve N_{i+1} as a function of N_i – $t_{i+1} - t_i = h$ and h can have values $h = 10^{-4} \ldots 0.1$. In other words solve N at t_i, $i = 2, \ldots, n$ $(N(t_1) = N(0))$ to some large n and plot it. Compare approximation with the excact solution $N(t) = N(0)e^{-\lambda t}$.
4. Object has acceleration $g(t)$ which varies in time:

$$a(t) = \frac{d^2x(t)}{dt^2} = g(t),$$

   from which

$$v(t) = \frac{dx(t)}{dt} = v(0) + \int_0^t a(t')dt',$$

and

$$x(t) = x(0) + \int_0^t v(t')dt'.$$

Let $g(t) = ct$, where $c = 0.1\,\mathrm{ms}^{-3}$, and initial conditions $x(0) = 0$, $v(0) = 1\,\mathrm{ms}^{-1}$. Let's denote $t_i = (i-1)h$, $i = 1, \ldots, n$ where $h = 0.01\,\mathrm{s}$. Calculate first the velocities $v(t_i)$ and then the location x of the object at time t_i. Use first the build-in function `Integrate` and then approximate integrals using trapezoidal rule.