ATKIV Numerical Programming
Exercise 4.3 Padé approximant

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Suppose our function was given in a more general form

$$g(x) \approx c_1 x + c_3 x^3 + c_5 x^5. \tag{1}$$

We wish to find it's Padé approximant

$$f(x) \approx R(x) = \frac{\sum_{k=0}^{N} a_k x^k}{1 + \sum_{k=1}^{M} b_k x^k}, \tag{2}$$

where the coefficients a_k and b_k are determined by the $M + N + 1$ equations

$$f(0) = R(0), \tag{3}$$

$$\frac{d^k f(x)}{dx^k}\Big|_{x=0} = \frac{d^k R(x)}{dx^k}\Big|_{x=0}, \tag{4}$$

where $k = 1, \ldots, M + N$. Using the fact that the function $g(x)$ is odd, we may infer that as the polynomial in the denominator cannot be odd, it must be even while the numerator is odd. Furthermore, from Eq. (1) we can calculate $g^{(k)}(0)$ for $k = 1, \ldots, 5$. So, we can choose $N = 3$ and $M = 2$. Thus, the approximant becomes

$$R(x) = \frac{a_1 x + a_3 x^3}{1 + b_2 x^2} \tag{5}$$

Rest is pure algebra. Here are the five derivatives of R:

$$\begin{aligned}
R'(x) &= \frac{a_1 - a_1 b_2 x^2 + a_3 x^2 \left(3 + b_2 x^2\right)}{\left(1 + b_2 x^2\right)^2} \\
R''(x) &= -2x \frac{\left(a_3 - a_1 b_2\right)\left(-3 + b_2 x^2\right)}{\left(1 + b_2 x^2\right)^3} \\
R^{(3)}(x) &= 6 \frac{\left(a_3 - a_1 b_2\right)\left(1 - 6 b_2 x^2 + {b_2}^2 x^4\right)}{\left(1 + b_2 x^2\right)^4} \\
R^{(4)}(x) &= 24x \frac{b_2 \left(-a_3 + a_1 b_2\right)\left(5 - 10 b_2 x^2 + {b_2}^2 x^4\right)}{\left(1 + b_2 x^2\right)^5} \\
R^{(5)}(x) &= -120 \frac{b_2 \left(-a_3 + a_1 b_2\right)\left(-1 + 15 b_2 x^2 - 15 {b_2}^2 x^4 + {b_2}^3 x^6\right)}{\left(1 + b_2 x^2\right)^6}
\end{aligned}$$

Derivatives of $g(x)$ are

$$
\begin{aligned}
g'(x) &= c_1 + 3c_3x^2 + 5c_5x^4, \\
g''(x) &= 6c_3x + 20c_5x^3, \\
g^{(3)}(x) &= 6c_3 + 60c_5x^2, \\
g^{(4)}(x) &= 120c_5x, \\
g^{(5)}(x) &= 120c_5.
\end{aligned}
$$

When we equate these at $x = 0$, we have the following set of equations

$$
\begin{aligned}
c_1 &= a_1, && (6) \\
c_3 &= a_3 - a_1b_2, && (7) \\
c_5 &= b_2\left(-a_3 + a_1b_2\right). && (8)
\end{aligned}
$$

This can be easily solved.

$$
\begin{aligned}
a_1 &= c_1, && (9) \\
b_2 &= -\frac{c_5}{c_1c_3}, && (10) \\
a_3 &= c_3 - \frac{c_5}{c_3c_1}. && (11)
\end{aligned}
$$

Plugging-in the given values for the coefficients, we get

$$
\begin{aligned}
a_1 &= 1 && (12) \\
b_2 &\approx -0.3994 && (13) \\
a_3 &\approx -0.0664 && (14)
\end{aligned}
$$

In Fig. 1 the function $R(x)$ is plotted against the $\tan(x)$, whose power series coefficients the c_k:s actually are.

Note, that it's very easy to derive $R(x)$ of the form in Eq. (5) for the tangent function:

$$
\begin{aligned}
\tan(x) &= \frac{x^2 - \pi^2/4}{x^2 - \pi^2/4}\tan(x) && (15) \\
&= \frac{(x^2 - \pi^2/4)(x - x^3/3 + \mathcal{O}(x^5))}{x^2 - \pi^2/4} && (16) \\
&= \frac{-\pi^2x/4 + (1 - \pi^2/12)x^3 + \mathcal{O}(x^5)}{x^2 - \pi^2/4} && (17) \\
&\approx \frac{x - (4/\pi^2 - 1/3)x^3}{1 - 4x^2/\pi^2} && (18)
\end{aligned}
$$

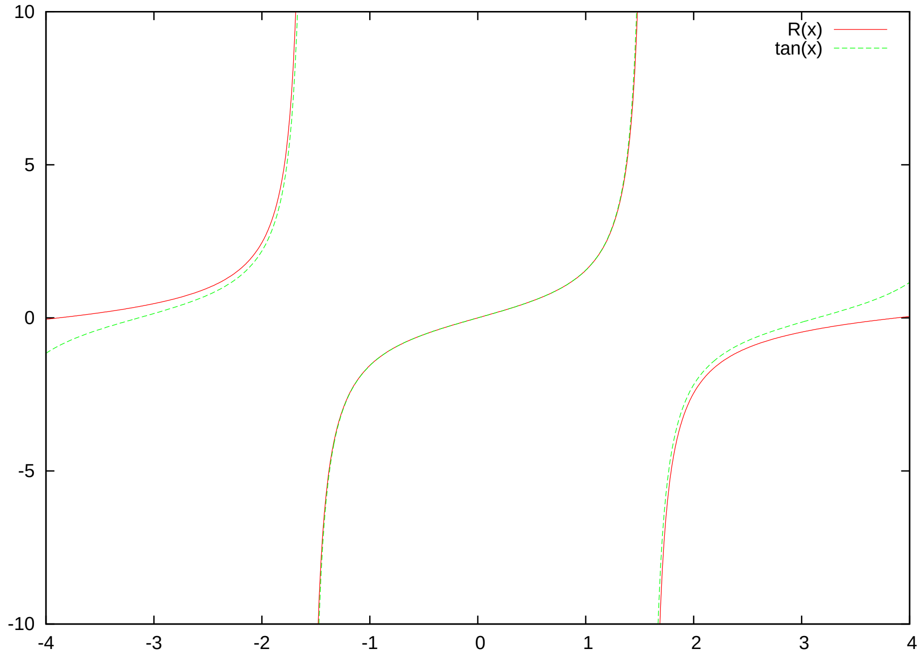

Figure 1: Padé approximant $R(x) \approx \tan(x)$.