1. **Roots and numerical integration.** Your task is to find the value of the integral

$$\int_{-1}^{1} \sqrt{1-x^2}\,\sin(\cos\pi x)\,dx \tag{1}$$

using Gaussian quadratures. The formula for this approximation is

$$\int_{a}^{b} W(x)f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i). \tag{2}$$

Here, $w_i$ are given weighting coefficients and values $x_i$ are roots of a polynomial that is orthogonal with respect to the weight function $W(x)$ on the interval $[a, b]$. In the specific case of $W(x) = \sqrt{1-x^2}$, it is the Chebyshev polynomial of the second kind, $U_n(x)$. It is orthogonal on the interval $[-1, 1]$ where it has exactly $n$ roots, and the associated weighting coefficients can be shown to be

$$w_i = \frac{\pi}{U_{n-1}(x_i)U_n'(x_i)}. \tag{3}$$

Of course, here function $f(x) = \sin(\cos\pi x)$.

1) To use Eq. (2), you must first find the roots of $U_n$ *numerically* on the given interval. You are free to choose whatever value of $n$ you wish, as long as it's sensible and justified. Use Numerical Recipes `zbrak` for initial bracketing, and `rtnewt` or `zbrent` for final root finding.
2) Naturally you need to be able to evaluate $U_n(x)$ and its derivative $U_n'(x)$. In order to do this, use recurrence relations (4,5).
3) When you have the roots $x_i$ ready, the coefficients $w_i$ can be computed, and the summation of Eq. (2) gives you the approximate value of the integral. Compare the result to the one given by Numerical Recipes function `qromb`.
4) Is using Gaussian quadratures beneficial or even sensible in this problem? Address this question in your report and give some explanation.

$$\begin{aligned} U_n(x) &= 2xU_{n-1}(x) - U_{n-2}(x), && (4)\\ U_n'(x) &= 2U_{n-1}(x) + 2xU_{n-1}'(x) - U_{n-2}'(x) && (5) \end{aligned}$$

$$\begin{matrix} U_0(x) = 1, & U_1(x) = 2x \\ U_0'(x) = 0, & U_1'(x) = 2 \end{matrix} \tag{6}$$