Introduction to Particle Physics Spring 2007 Problem set 4

1. An example of a simple group is U(1), group of 1-dim. unitary "matrices" which rotate the phase of complex numbers. It is defined by $z' = u\,z$, $z \in C$ and $|z'| = |z|$. Thus, the fundamental representation can be taken to be
$$u = e^{-i\theta}, \qquad 0 \le \theta < 2\pi.$$
Show that U(1) is a group and Abelian. What is the "generator"?
Show that also $u_n = e^{-in\theta}$, n integer, is a (1-dimensional) representation of the group, and if n is *not* an integer this is not a representation. (Hint: remeber the 2π-periodicity of θ!). Also show that the only faithful reps are the fundamental (u_1) and the conjugate (u_{-1}) one.
U(1) also has 2-dimensional representations. Construct one by considering the real and imaginary parts of the definition separately, i.e.
$$\begin{pmatrix} \mathrm{Re}\, z' \\ \mathrm{Im}\, z' \end{pmatrix} = U \begin{pmatrix} \mathrm{Re}\, z \\ \mathrm{Im}\, z \end{pmatrix}.$$
2x2-matrices U form a group SO(2), which is equivalent to U(1).

2. When the pion energy in lab frame is $\sim 300\,\mathrm{MeV}$ the pion-nucleon scattering proceeds predominantly through the $\Delta(1232)$ resonance:
$$\pi + N \to \Delta \to \pi + N'$$
where $N, N' = \mathrm{p}, \mathrm{n}$. This proceeds through strong interactions. Using the isospin symmetry, show that the ratios of the cross-sections of the processes
$$\pi^+ + \mathrm{p} \to \pi^+ + \mathrm{p}, \qquad \pi^- + \mathrm{p} \to \pi^0 + \mathrm{n}, \qquad \pi^- + \mathrm{p} \to \pi^- + \mathrm{p}$$
are 9:2:1.

3. Using the definition of SU(N) ($U^\dagger = U^{-1}$, $\det U = 1$), show that the SU(N) matrices can be written as $U = e^{-iA}$, where A is traceless and Hermitean ($A^\dagger = A$). Show that A has $N^2 - 1$ independent real degrees of freedom. Thus, we can choose matrices λ_k such that
$$A = \sum_{k=1}^{N^2-1} \theta_k \lambda_k, \qquad \theta_k \in R,$$
where λ_k satisfy the conditions
$$\lambda_k^\dagger = \lambda_k, \qquad \mathrm{Tr}\,\lambda_k = 0, \qquad \mathrm{Tr}\,\lambda_i \lambda_j = \frac{1}{2}\delta_{ij} \ \ \text{(orthogonality)}.$$
Show that the generators obey the algebra
$$[\lambda_i, \lambda_j] = i \sum_k f_{ijk} \lambda_k$$
for some $f_{ijk} \in R$. These are called *structure constants* of the group. (Hint: show that the commutator is traceless and antihermitean). Show that f_{ijk} is fully antisymmetric, i.e. it changes sign if any two indices are swapped. (Hint: multiply above eqn by λ_ℓ and take trace.)