1. Assuming that ϕ is a solution of Klein-Gordon wave equation $[\partial_0^2 - \nabla^2 + m^2]\phi = 0$, show that the charge

$$Q = i \int d^3\mathbf{x} \, (\phi^* \partial_0 \phi - \phi \partial_0 \phi^*)$$

is conserved. (Hint: assume $\phi(x \to \infty) = 0$)

2. Let us now "2nd quantise" the fields $\phi \to \hat{\phi}$, $\phi^* \to \hat{\phi}^\dagger$, where

$$\begin{aligned} \hat{\phi} &= \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^3 2E_\mathbf{p}}} \left[\hat{a}_\mathbf{p} e^{-ip\cdot x} + \hat{b}_\mathbf{p}^\dagger e^{ip\cdot x} \right] \\ \hat{\phi}^\dagger &= \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^3 2E_\mathbf{p}}} \left[\hat{a}_\mathbf{p}^\dagger e^{ip\cdot x} + \hat{b}_\mathbf{p} e^{-ip\cdot x} \right] \end{aligned}$$

where $p = (E_\mathbf{p}, \mathbf{p})$, $E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}$. Using commutation relations

$$[\hat{a}_\mathbf{p}, \hat{a}_\mathbf{q}^\dagger] = [\hat{b}_\mathbf{p}, \hat{b}_\mathbf{q}^\dagger] = \delta^{(3)}(\mathbf{p} - \mathbf{q})$$

show that the charge in 1) becomes the operator

$$\hat{Q} = \int d^3\mathbf{p} \left[\frac{1}{2}(\hat{a}_\mathbf{p}^\dagger \hat{a}_\mathbf{p} + \hat{a}_\mathbf{p} \hat{a}_\mathbf{p}^\dagger) - \frac{1}{2}(\hat{b}_\mathbf{p}^\dagger \hat{b}_\mathbf{p} + \hat{b}_\mathbf{p} \hat{b}_\mathbf{p}^\dagger) \right] = \int d^3\mathbf{p} \left[\hat{a}_\mathbf{p}^\dagger \hat{a}_\mathbf{p} - \hat{b}_\mathbf{p}^\dagger \hat{b}_\mathbf{p} \right].$$

3. An eigenstate of momentum is $|\phi_\mathbf{p}\rangle \equiv \hat{a}_\mathbf{p}^\dagger |0\rangle$, where $|0\rangle$ is the vacuum state, normalised as $\langle 0|0\rangle = 1$. Show that the momentum states obey

$$\langle \phi_\mathbf{q} | \phi_\mathbf{p} \rangle = \delta^{(3)}(\mathbf{p} - \mathbf{q})$$

4. The Hamilton operator for free complex Klein-Gordon field is

$$\hat{H} = \int d^\mathbf{x} \left[\partial_0 \hat{\phi} \partial_0 \hat{\phi}^\dagger + \nabla \hat{\phi}^\dagger \cdot \nabla \hat{\phi} + m^2 \hat{\phi}^\dagger \hat{\phi} \right].$$

Show that this can be written as

$$\hat{H} = \int d^3\mathbf{p} E_\mathbf{p} \left[\hat{a}_\mathbf{p}^\dagger \hat{a}_\mathbf{p} + \hat{b}_\mathbf{p}^\dagger \hat{b}_\mathbf{p} + \delta^{(3)}(\mathbf{0}) \right].$$

5. Show that when the volume V of the system is finite,

$$\delta^{(3)}(\mathbf{0}) \to \frac{V}{(2\pi)^3}$$

(Hint: consider definition of δ as an integral over $\mathbf{x}$). How would you interpret the last part of $\hat{H}$ in (4)?