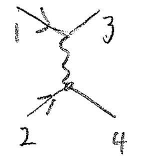
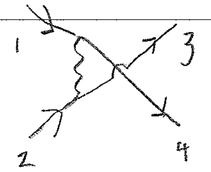
On the other hand, Møller scattering
$e^- + e^- \to e^- + e^-$ gives



$$M_1 = \frac{-e^2}{(q_1 - p_3)^2} \bar{u}_3 \gamma^\mu u_1 \; \bar{u}_4 \gamma_\mu u_2$$



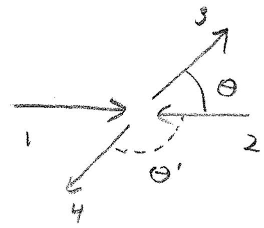
$$M_2 = \frac{-e^2}{(q_1 - p_4)^2} \bar{u}_4 \gamma^\mu u_1 \; \bar{u}_3 \gamma_\mu u_2 \times (-1)$$

$|M_1|^2$ gives result equivalent to $e\mu$-scattering

$|M_2|^2$ too, but $p_3 \leftrightarrow p_4$:

$\to \theta \to \theta' = \pi - \theta$

or $\cos\theta \to -\cos\theta$



Interference $M_1 M_2^* + M_2 M_1^*$ gives (when $\frac{1}{4}\sum_{s_1 s_2 s_3 s_4}$)

$$\langle (M_1 M_2^* + M_1^* M_2) \rangle = \frac{e^4}{(q_1 - p_3)^2 (q_1 - p_4)^2} \times$$

$$\left(\frac{1}{4} \sum_s [\bar{u}_3 \gamma^\mu u_1][\bar{u}_4 \gamma^\nu u_1]^+ [\bar{u}_4 \gamma_\mu u_2][\bar{u}_3 \gamma_\nu u_2]^+ + h.c. \right)$$

$$= \frac{e^4}{(q_1 - p_3)^2 (q_1 - p_4)^2} \frac{1}{4} \sum_s \bar{u}_3 \gamma^\mu \underbrace{u_1 \bar{u}_1}_{\not{q}_1 + m} \gamma^\nu \underbrace{u_4 \bar{u}_4}_{\not{p}_4 + m} \gamma_\mu \underbrace{u_2 \bar{u}_2}_{\not{q}_2 + m} \gamma_\nu u_3 + h.c.$$

$$= \frac{e^4}{(q_1 - p_3)^2 (q_1 - p_4)^2} \frac{1}{4} \mathrm{Tr}\left[\gamma^\mu (\not{q}_1 + m) \gamma^\nu (\not{p}_4 + m) \gamma_\mu (\not{q}_2 + m) \gamma_\nu (\not{p}_3 + m) \right] + h.c.$$

The trace can be in principle evaluated in straightforward γ-matrix algebra. This is tedious!

* Taking non-relativistic limit, $|\vec{p}_1| \sim |\vec{q}_1| \ll m$, Trace becomes $m^4 \mathrm{Tr}[\underbrace{\gamma^\mu \gamma^\nu \gamma_\mu}_{-2\gamma^\nu} \gamma_\nu] = -m^4 \cdot 2 \cdot 4 \cdot 4$

Now $(q_1 - p_3)^2 = (\bar{q}_1 - \bar{p}_3)^2 = 2|\bar{q}_1|^2(1-\cos\theta)$

$(q_1 - p_4)^2 = (\bar{q}_1 - \bar{p}_4)^2 = 2|\bar{q}_1|^2(1+\cos\theta)$

Thus, $\langle |M|^2 \rangle = \langle |M_1|^2 \rangle + \langle |M_2|^2 \rangle - \langle 2|M_1 M_2^*| \rangle$

$$= \frac{4e^4}{|\bar{q}_1|^4(1-\cos\theta)^2} + \frac{4e^4}{|\bar{q}_1|^4(1+\cos\theta)^2} - \frac{4e^4}{|\bar{q}_1|^4(1-\cos^2\theta)}$$

pg. 137

Feynman rule ⑥!

* In ultrarelativistic limit we set $m = 0$. Now we need to evaluate trace. Helpful properties:

$\gamma_\mu \not{a} \gamma^\mu = -2\not{a}$; $\gamma_\mu \not{a}\not{b} \gamma^\mu = 4\, a \cdot b$; $\gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu = -2\not{c}\not{b}\not{a}$

We will not do the calculation here, final result is

$$\langle |M|^2 \rangle = 2e^4 \left\{ \frac{4 + (1+\cos\theta)^2}{(1-\cos\theta)^2} + \frac{4 + (1-\cos\theta)^2}{(1+\cos\theta)^2} - \frac{2}{1-\cos^2\theta} \right\}$$

$$= 2e^4 \left\{ \frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} - \frac{2s^2}{tu} \right\}$$

with Mandelstam variables

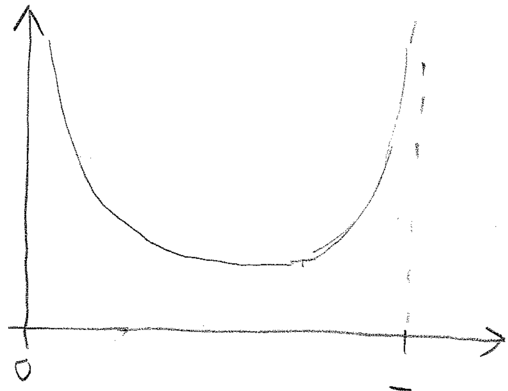
$s = (q_1 + q_2)^2 = 4E^2$

$t = (q_1 - p_3)^2 = -2E^2(1-\cos\theta)$

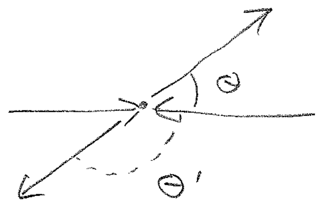
$u = (q_1 - p_4)^2 = -2E^2(1+\cos\theta)$

* In both limits, $\langle |M|^2 \rangle$ looks like the following:

- symmetric w.r.t. $\pi/2$
- diverges at $\theta = 0, \pi$
  = forward & backward peaks



- Symmetric because we do not know which electron is which!



QED:

- Perhaps the most accurately verified theory in physics!
- Describes <u>all</u> electromagnetic interactions
  (albeit for non-relativistic situations Schrödinger equation is much easier to use)
- Expansions converge very well; QED "solved"
- Difficult to use for bound states (atom).
- At fundamental level there are some problems of theoretical nature. (QFT lectures)

9. Strong interactions

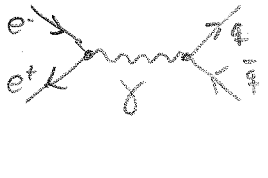
- Very complicated phenomenology! (Colour/quark confinement, → hadrons
- At fundamental level looks quite similar to QED. Why different phenomenology?
  - Gluons self-interact
  - Strong coupling
- We'll discuss QCD here in quite superficial manner.

9.1. Quark model [Gell-Mann, Ne'eman, Zweig 1959-64]

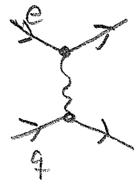
- As described in section Flavour symmetries quark model describes hadron (meson, baryon) classification very well (Isospin, flavour SU(3) etc.)
- "Eightfold way"; octets, decuplets etc..
- Explanation: quarks have colour but only colour neutral states exist (does not explain why)

9.2. Hadron production in e^+e^--collisions

- We want to study the inner structure of hadrons. Because quarks are electrically charged, we can use well-known QED processes to study this:



$q\bar{q}$-production

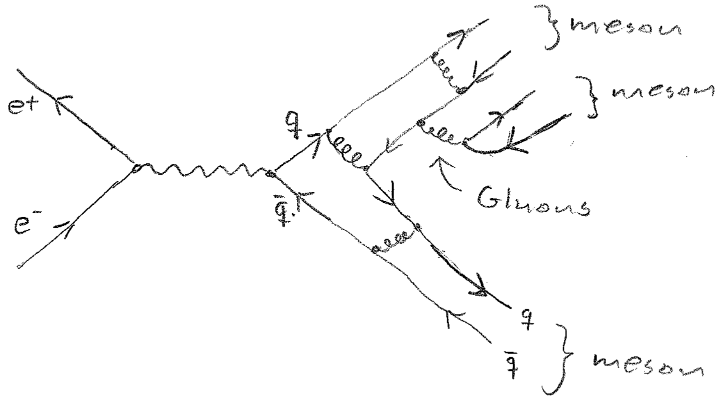


eq-scattering

- These are pure QED!

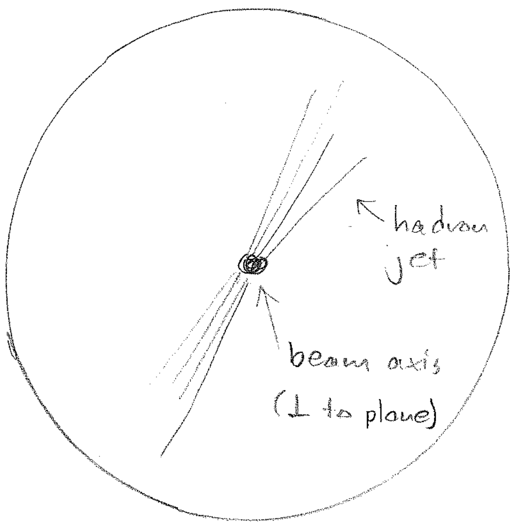
- What happens in $e^+e^- \to q\bar{q}$ collision?

  [SLAC, Stanford ; LEP, Cern ; planned linear collider]



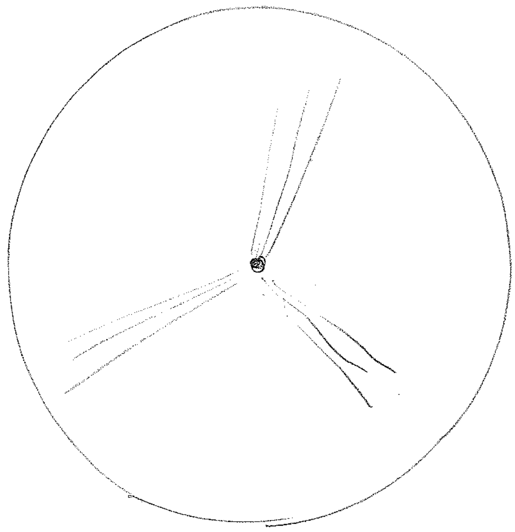
possibly 10's or 100's of hadrons (mostly mesons)

- Experimental signature : hadron jets :



2-jet event

$e^+e^- \to q\bar{q} \to$ hadrons

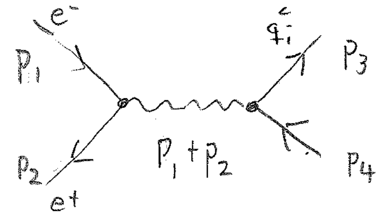


3-jet event

$e^+e^- \to q\bar{q}\,G \to$ hadrons

$\uparrow$ gluon

- The $q\bar{q}$-production is pure QED which we know how to calculate:



q_i^c : quark of flavour i (u,d,s...) color c (R,G,B)

E = e^+ (or e^-) energy in center of mass system

Q_i = charge of quark i in units of e $\quad (Q_i e,\ Q_i = \pm\frac{1}{3}, \pm\frac{2}{3})$

$$\mathcal{M} = \frac{(-ig_{\mu\nu})}{(p_1+p_2)^2}[\bar{v}_2\, i(-e)\gamma^\mu u_1][\bar{u}_3\, iQ_i e\gamma^\nu v_4] \quad (i)$$

$$= \frac{Q_i e^2}{(p_1+p_2)^2}[\bar{v}_2\gamma^\mu u_1][\bar{u}_3\gamma_\mu v_4] \quad \Rightarrow \quad \text{sum over spins}$$

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{4}\frac{Q_i^2 e^4}{(p_1+p_2)^4}\mathrm{Tr}[\gamma^\mu \overbrace{(\not{p}_1+m_e)}^{u_1\bar{u}_1}\gamma^\nu \overbrace{(\not{p}_2-m_e)}^{v_2\bar{v}_2}] \times$$
$$\mathrm{Tr}[\gamma_\mu(\not{p}_4-m_q)\gamma_\nu(\not{p}_3+m_q)]$$

Now $\mathrm{Tr}[\gamma^\mu \not{a}\gamma^\nu \not{b}] = 4[a^\mu b^\nu + a^\nu b^\mu - g^{\mu\nu}a\cdot b]$ (pg. 136)

$\Rightarrow$

$$\langle|\mathcal{M}^2|\rangle = 4\frac{Q_i^2 e^4}{(p_1+p_2)^4}[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu}(p_1\cdot p_2 + m_e^2)] \times$$
$$[p_{4,\mu}p_{3,\nu} + p_{4,\nu}p_{3,\mu} - g_{\mu\nu}(p_3\cdot p_4 + m_q^2)]$$

$$= 8\frac{Q_i^2 e^4}{(p_1+p_2)^4}\Big(p_1\cdot p_4\ p_2\cdot p_3 + p_1\cdot p_3\ p_2\cdot p_4$$
$$+ p_1\cdot p_2\, m_q^2 + p_3\cdot p_4\, m_e^2 + 2m_q^2 m_e^2\Big)$$

$$p_1 \cdot p_4 = p_2 \cdot p_3 = E^2 - \bar{p}_1 \cdot \bar{p}_4 = E^2 + |\bar{p}_1||\bar{p}_4| \cos\theta$$

$$p_1 \cdot p_3 = p_2 \cdot p_4 = E^2 - |\bar{p}_1||\bar{p}_4| \cos\theta$$

$$(p_1+p_2)^2 = 4E^2 \quad ; \quad p_1 \cdot p_2 = E^2 + |\bar{p}_1|^2 = 2E^2 - m_e^2 \; ; \; p_3 \cdot p_4 = 2E^2 - m_q^2$$

$$p_1 \cdot p_4 \; p_2 \cdot p_3 + p_1 \cdot p_3 \; p_2 \cdot p_4 = 2E^4 + 2 \underbrace{|\bar{p}_1|^2}_{= E^2 - m_e^2} \underbrace{|\bar{p}_4|^2}_{= E^2 - m_q^2} \cos^2\theta =$$

$$\Rightarrow \langle |M|^2 \rangle = Q_i^2 e^4 \left[1 + \frac{m_q^2 + m_e^2}{E^2} + \left(1 - \frac{m_e^2}{E^2}\right)\left(1 - \frac{m_q^2}{E^2}\right) \cos^2\theta \right]$$

Finally, total σ we obtain from page 122; (123)
integrating over $\int d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta$ (Exercise)

$$\underline{\sigma_{tot}^{q_i c} = Q_i^2 \frac{\pi}{3} \left(\frac{\alpha}{E}\right)^2 \sqrt{\frac{1 - m_q^2/E}{1 - m_e^2/E}} \left(1 + \frac{m_q^2}{2E^2}\right)\left(1 + \frac{m_e^2}{2E^2}\right) \theta(E - m_q)}$$

The total cross-section into any quarks is

$$\sigma_{tot} = \sum_{\substack{\text{flavour } i \\ \text{colour } c}}' \sigma_{tot}^{q_i, c} = N_c \sum_i' \sigma_{tot}^{q_i} \qquad (N_c = 3)$$

If $E \gg m_q \gg m_e$, this simplifies to

$$\underline{\sigma_{tot} \approx N_c \sum_i' Q_i^2 \frac{\pi}{3} \left(\frac{\alpha}{E}\right)^2}$$

only flavours where $m_{q_i} < E$

This threshold effect is clearly demonstrated in the ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

exactly as in page 146, with $Q=1$, $m_q \to m_\mu$

Thus, using the approximate formula,

$$\boxed{R(E) = N_c \sum_{\substack{i \\ m_i < E}} Q_i^2}$$

$N_c = 3$ for QCD ($SU(3)$)

Thus, if $m_s < E < m_c$, u, d, s contribute, $R = 3\cdot\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right) = \underline{2}$

$m_c < E < m_b$, $R = 2 + 3\cdot\left(\frac{2}{3}\right)^2 = \underline{\frac{10}{3}}$

$m_b < E < m_t$, $R = \frac{11}{3}$

$m_t < E$, $R = \frac{15}{3} = 5$ (not reached experimentally)

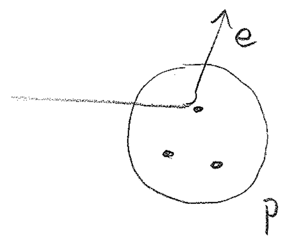
Works very well! verification of $N_c = 3$ and m_q.

(Does not work for small $E \lesssim m_s$: then there are no hadrons which are light enough to be produced! In that case we cannot study the EM process separately from QCD)

9.3 Deep inelastic $e^- p^+$ - scattering

[HERA/DESY ; eRHIC(?)]

- Precision probe for proton (nucleon) structure!



* High-energy electron

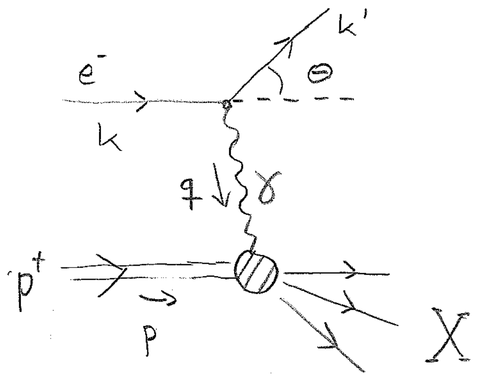
$\updownarrow$

high resolution

* Large-angle scattering
→ inner structure! (Rutherford: atomic nucleus)

* Proton splinters into numerous hadrons = inelastic

* Kinematics



Inclusive process: only scattered e^- is identified (and its $k' = (E', \vec{k})$ measured)
⇒ two parameters measured,

$k'^0 = E'$ and $\cos\theta$

(ϕ-angle has no significance)

(in elastic scattering $e^- p^+ \to e^- p^+$ there is only one variable, say $\cos\theta$, because $(k-k'+P)^2 = m_p^2$)

Instead of E', $\cos\theta$, commonly used variables are

$$Q^2 \equiv -q^2 = (k-k')^2 \quad ; \quad x \equiv \frac{Q^2}{2q\cdot P} \quad \text{"Bjorken } x\text{"}$$

← proton 4-momentum

1-to-1 mapping between $(E', \cos\theta) \leftrightarrow (Q^2, x)$ (Exercise)

* If $E \gg m_e$, cross-section is commonly parametised as

$$\frac{d\sigma}{dE' \, d\Omega} = \left(\frac{\alpha}{2E\sin^2\frac{\theta}{2}}\right)^2 \left[2W_1(Q^2,x)\sin^2\frac{\theta}{2} + W_2(Q^2,x)\cos^2\frac{\theta}{2}\right]$$

↑

energy and angle

* W_1 and W_2 are called <u>structure functions</u>.

These encode our knowlede of the proton structure, and are experimentally well-known

↑ at high energy

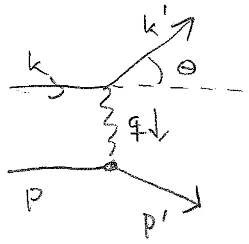
* Comparing to <u>elastic</u> Mott scattering (page 138):

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{2E^2\sin^2\frac{\theta}{2}}\right)^2 E^2\cos^2\frac{\theta}{2} \qquad (E \gg m_e;\ |\vec{q}| = E)$$

Here E' is not independent of E, θ. Recall that here we assumed m_μ (or m_p) $\gg E$. If we don't do that, scattering cross-section is (start from page 137)

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha Q_i}{4ME\sin^2\frac{\theta}{2}}\right)^2 \left(2Q^2\sin^2\frac{\theta}{2} + 4M^2\cos^2\frac{\theta}{2}\right) \qquad (*)$$

where myons are generalised to particle of mass $= M$, charge $= Q_i e$ (m_e neglected)



$$Q^2 = -q^2 = -(k-k')^2 = +2E^2(1-\cos\theta)$$
$$= 4E^2\sin^2\frac{\theta}{2}$$

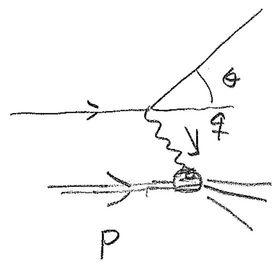
9.4. Parton model (Bjorken, Callan, Gross 1967-69)

* Bjorken predicted scaling relations for structure functions:

$$m_p W_1(Q^2, x) \to F_1(x)$$

$$\frac{Q^2}{2m_p x} W_2(Q^2, x) \to F_2(x)$$

These should be valid when Q^2 and $q \cdot p$ are large and x is small
= "deep inelastic" limit



* In 1969 Callan and Gross suggested that F_1 and F_2 are related:

$$2x F_1(x) = F_2(x)$$

Thus, we went from 2 functions of 2 variables to 1 function of 1 variable!
These relations arise from Parton model: high-energy proton consists of ~free particles, "partons". (These are quarks & gluons - however, quark model at the time was problematic and gluons unknown)