

1. Let us consider a two-state system. We denote the eigenstates of the Hamiltonian by $|\psi_i\rangle$ and the corresponding eigenvalues by $\hbar\omega_i$. That is,

$$H|\psi_i\rangle = \hbar\omega_i|\psi_i\rangle, \quad \text{where } i \in \{1, 2\}. \quad (1)$$

Furthermore, let us denote the state of the system at time $t = t_0$ by

$$|\Psi(t_0)\rangle = c_1(t_0)|\psi_1\rangle + c_2(t_0)|\psi_2\rangle. \quad (2)$$

We want to measure a dynamical quantity β to which corresponds a Hermitian operator B . We denote the eigenstates of B by $|u_j\rangle$ and the corresponding eigenvalues by b_j . That is,

$$B|u_j\rangle = b_j|u_j\rangle, \quad \text{where } j \in \{1, 2\}.$$

In addition, we assume that the eigenvalues of both H and B are non-degenerate (i.e. $\omega_1 \neq \omega_2$ and $b_1 \neq b_2$).

- a) Let us assume that B and H commute. What are the probabilities for obtaining b_1 and b_2 when we measure the dynamical quantity β at time $t \neq t_0$? Do these probabilities depend on time?

Note that it follows from our assumptions that B and H have the same eigenstates.

- b) Like a) but now B and H do not commute.

Note the similarity with exercise 6 problem 5. For a discussion on the two-state system, see e.g. Wikipedia, two-state system.

2. The first excited state of an iron atom ^{53}Fe collapses to the ground state by emitting a photon of energy 14.4 keV. The lifetime of the first excited state is 141 ns.

- a) Estimate the linewidth ΔE of the spectral line.

- b) Calculate the recoil energy of the atom.

<http://www.webelements.com/webelements/elements/text/Fe/radio.html>

3. Prove the following commutation relations

a) $[A, B] = -[B, A]$

b) $[A, (B + C)] = [A, B] + [A, C]$

c) $[A, BC] = [A, B]C + B[A, C]$

d) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

e) $[A, B]^\dagger = [B^\dagger, A^\dagger]$

Show that a)-c) hold also for Poisson brackets.

4. Show that the operators corresponding to the Cartesian components of the angular momentum satisfy the commutation relation

$$[L_x, L_y] = i\hbar L_z.$$

Show that the indices x, y and z in the above equation can be cyclically permuted ($x \rightarrow y \rightarrow z \rightarrow x$). In addition, calculate $\mathbf{L} \times \mathbf{L}$, where \mathbf{L} is the angular momentum operator.

5. Show that the probability $P(b_m, t)$ is a Gaussian function when the energy distribution function $c(E)$ is such. Calculate the mean square deviations of time and energy using these distributions and show that

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

This problem is related to the discussion given on the lecture notes p. 85.