## Instructions on how to solve exercise 14 problem 2

Consider a dipole transition in the Lyman series. Show that the matrix element of the $z$ component of the dipole operator is (when we omit the elementary charge)

$$\langle \psi_{100}|z|\psi_{n10}\rangle = \frac{16}{\sqrt{3}} n^{7/2}(n-1)^{n-5/2}(n+1)^{-n-5/2}.$$

### Instructions

For a general discussion on the Lyman series, see e.g. Wikipedia, Lyman series. What comes to the problem at hand, one begins by briefly explaining why the matrix element $\langle \psi_{100}|z|\psi_{nlm}\rangle$ vanishes if $l \neq 1$ or $m \neq 0$. Then one takes the matrix element $\langle \psi_{100}|z|\psi_{n10}\rangle$ under scrutiny. One obtains

$$\begin{aligned}\langle \psi_{100}|z|\psi_{n10}\rangle &= N_{10}N_{n1}\int_0^\infty r^4 e^{-(1+\frac{1}{n})r} L_0^1(2r) L_{n-2}^3\left(\frac{2r}{n}\right) dr \\ &\times \int_0^\pi \int_0^{2\pi} \sin\theta \cos\theta Y_0^0(\theta,\varphi)^* Y_0^1(\theta,\varphi)\, d\varphi d\theta. \end{aligned} \tag{1}$$

One calculates the angular integral and substitutes the associated Laguerre polynomial of order one. One obtains

$$\langle \psi_{100}|z|\psi_{n10}\rangle = \frac{N_{10}N_{n1}}{\sqrt{3}} \int_0^\infty r^4 e^{-(1+\frac{1}{n})r} L_{n-2}^3\left(\frac{2r}{n}\right) dr. \tag{2}$$

A certain change of variables yields

$$\langle \psi_{100}|z|\psi_{n10}\rangle = N_I \int_0^\infty e^{-t} t^3 e^{-\frac{n-1}{2}t} t L_{n-2}^3(t)\, dt, \tag{3}$$

where $N_I = \frac{N_{10}N_{n1}}{\sqrt{3}}\left(\frac{n}{2}\right)^5$. One expresses the factor $e^{-\frac{n-1}{2}t}$ in terms of the associated Laguerre polynomials by employing the generating function. After that, by employing a recurrence relation, one writes the factor $tL_{n-2}^3(t)$ as a linear combination of the associated Laguerre polynomials. The integral can now be calculated by employing the orthogonality property of the associated Laguerre polynomials.