

1. The electron of a hydrogen atom is in the $4f$ state. To be more precise, its wave function is $\psi_{431}(r, \theta, \varphi) = R_{43}(r)Y_1^3(\theta, \varphi)$. Find the eigenvalues of the operators H , L^2 and L_z in this state.
2. Let us assume that a hydrogen atom is in a $3p$ state. Show that the radial part of its wave function is

$$\frac{1}{r}u_{31}(r) = \frac{4}{81\sqrt{6}} e^{-\frac{r}{3}} r(6-r).$$

3. The solutions of a differential equation

$$\frac{d^2w}{dr^2} - 2\sqrt{-E} \frac{dw}{dr} + \left(\frac{2}{r} - \frac{\ell(\ell+1)}{r^2} \right) w = 0,$$

are polynomials

$$w(r) = \sum_{k=\ell+1}^{\infty} a_k r^k.$$

Show that the coefficients of the polynomial satisfy the recurrence relation

$$a_{k+1} = 2 \frac{k\sqrt{-E} - 1}{k(k+1) - \ell(\ell+1)} a_k.$$

4. Calculate the normalization factor N_{nl} for the Hydrogen atom.

$$I = N_{nl}^2 \int_0^{\infty} r^{2l+2} \exp^{-\frac{2r}{n}} \left[L_{n-l-1}^{2l+1} \left(\frac{2r}{n} \right) \right]^2 dr.$$

Use the recurrence relation

$$tL_k^\alpha = (2k + \alpha + 1)L_k^\alpha - (k+1)L_{k+1}^\alpha - (k+\alpha)L_{k-1}^\alpha.$$

5. Calculate the expectation value of potential energy for the hydrogen atom and express the result in natural units. Calculate also $\langle v^2 \rangle$ using $\langle V \rangle + \langle T \rangle = E_n$. Finally, calculate the speed $v = \sqrt{\langle v^2 \rangle}$ in SI-units.

6. Determine the energy levels of a two-dimensional hydrogen atom. That is, determine the energy levels of a two-dimensional system comprising a proton and an electron. Note that the potential function of the system is

$$V(r) = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{r},$$

where r is the distance between the proton and the electron.