1. The wave function of a particle in one dimension is

$$\Psi(x,t) = xe^{-2|x|-i\omega t}.$$

a) Calculate the probability density.
b) Let us consider intervals of the form $[x - \frac{1}{4},\, x + \frac{1}{4}]$, where $x \in \mathbb{R}$. Determine the interval in which the particle is most likely found.
c) What is the probability to find the particle at the range $[-2, 2]$?
Part b) is the most challenging.

2. A particle is described by the wave function

$$\Psi(r) = \frac{e^{(ik-a)r}}{r}; \qquad a > 0 \text{ constant.}$$

Calculate the probability current density $\mathbf{S}$, when $r^2 = x^2 + y^2 + z^2$. How does $\mathbf{S}$ behave for large values of r?

3. A particle is described by the wave function

$$\Psi(\mathbf{r},t) = \frac{1}{N} e^{-2br} e^{-i\omega t}; \qquad b > 0 \text{ constant.}$$

First, calculate the normalization factor N so that

$$\int |\Psi(\mathbf{r},t)|^2 \, dV = 1.$$

Then, calculate the expectation values of momentum and energy.
The expectation value of momentum vanishes.

4. Let us consider a free particle in three dimensions. Show that the divergence of the probability current density vanishes. That is, show that

$$\nabla \cdot \mathbf{S} = 0.$$

What does this result mean?

5. Construct the general solution of the one-dimensional free particle Schrödinger equation. That is, construct the general solution of

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = i\hbar\frac{\partial\psi}{\partial t}.$$

Determine the probability density and the probability current density under the condition that the two arbitrary constants appearing in the general solution are real. Explain why the probability density depends on position. In general, if we know that the probability density is time-independent, what can we say about the probability current density?