1. Show that the operators
   a) $p = -i\hbar\nabla$,
   b) $H = -\frac{\hbar^2}{2m}\nabla^2 + V$, where V is a real function,
   c) x, the position operator,
   are Hermitian.

2. Show that a linear combination of two square-integrable, complex-valued functions is also square-integrable.

3. a) Calculate the density $\rho(k_F)$ of the one-dimensional Fermi gas as a function of the Fermi wave vector k_F.
   b) Calculate the density of states $g(E)$ as a function of energy E in one and two dimensions.

4. **a)** Show that a wave packet $\psi(x,t)$, with a weight function $g(k)$ of Gaussian shape, centered around $k = k_0$, is at time $t = 0$ itself a Gaussian function, centered at $x = 0$:
$$\psi(x,0) = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{ik_0 x} e^{-x^2/a^2}.$$
   The constant a describes the spread of the Gaussian weight function. The rest of the constants are due to normalization.

   b) A particle is described by a Gaussian wave packet. Is the particle then in a momentum eigenstate? Is it in an energy eigenstate of a free particle? Justify your answers.

5. Let us consider the waves
$$\psi_1(x,t) = \exp\left[i\left(\left(k_0 - \frac{\Delta k}{2}\right)x - \left(\omega_0 - \frac{\Delta\omega}{2}\right)t\right)\right],$$
$$\psi_2(x,t) = \exp\left[i\left(\left(k_0 + \frac{\Delta k}{2}\right)x - \left(\omega_0 + \frac{\Delta\omega}{2}\right)t\right)\right].$$
   a) At what velocities do these waves travel?

   Furthermore, let us define
$$\Psi(x,t) = \psi_1(x,t) + \psi_2(x,t) = 2e^{i(k_0 x - \omega_0 t)} \cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right).$$
   Then
$$|\Psi(x,t)|^2 = 4\cos^2\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right).$$
   b) What is the velocity of the maximum located at $x = 0$ when $t = 0$?