1. Let us consider a two-state system. We denote the eigenstates of the Hamiltonian by $|\psi_i\rangle$ and the corresponding eigenvalues by $\hbar\omega_i$. That is,

$$H|\psi_i\rangle = \hbar\omega_i|\psi_i\rangle, \quad \text{where } i \in \{1,2\}. \tag{1}$$

Furthermore, let us denote the state of the system at time $t = t_0$ by

$$|\Psi(t_0)\rangle = c_1(t_0)|\psi_1\rangle + c_2(t_0)|\psi_2\rangle. \tag{2}$$

We want to measure a dynamical quantity β to which corresponds a Hermitian operator B. We denote the eigenstates of B by $|u_j\rangle$ and the corresponding eigenvalues by b_j. That is,

$$B|u_j\rangle = b_j|u_j\rangle, \quad \text{where } j \in \{1,2\}. \tag{3}$$

In addition, we assume that the eigenvalues of both H and B are nondegenerate (i.e. $\omega_1 \neq \omega_2$ and $b_1 \neq b_2$).

a) Let us assume that B and H commute. What are the probabilities for obtaining b_1 and b_2 when we measure the dynamical quantity β at time $t \neq t_0$? Do these probabilities depend on time?
Note that it follows from our assumptions that B and H have the same eigenstates.

b) Like a) but now B and H do not commute.
For a discussion on the two-state system, see e.g. Wikipedia, two-state system.

2. Let us consider a Gaussian wave packet at a such a moment that the position uncertainty Δx is at minimum. Show that

$$\Delta x \Delta p = \frac{\hbar}{2}.$$

How do the values of Δx and Δp change when time elapses? Briefly explain the behavior of Δp.

3. The first excited state of an iron atom ^{53}Fe collapses to the ground state by emitting a photon of energy 14.4 keV. The lifetime of the first excited state is 141 ns.
a) Estimate the linewidth ΔE of the spectral line.
b) Calculate the recoil energy of the atom.
The mass of Iron-53 is given e.g. at http://en.wikipedia.org/wiki/Iron-57.

4. Prove the following commutation relations
   a) $[A, B] = -[B, A]$
   b) $[A, (B + C)] = [A, B] + [A, C]$
   c) $[A, BC] = [A, B]C + B[A, C]$
   d) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$
   e) $[A, B]^\dagger = [B^\dagger, A^\dagger]$
   Show that a)-c) hold also for Poisson brackets.

5. Show that the operators corresponding to the Cartesian components of the angular momentum satisfy the commutation relation

$$[L_x, L_y] = i\hbar L_z.$$

   Show that the indices x, y and z in the above equation can be cyclically permuted ($x \to y \to z \to x$). In addition, calculate $\mathbf{L} \times \mathbf{L}$, where $\mathbf{L}$ is the angular momentum operator.

6. Show that the probability $P(b_m, t)$ is a Gaussian function when the energy distribution function $c(E)$ is such. Calculate the mean square deviations of time and energy using these distributions and show that

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

   This problem is related to the discussion given on the lecture notes p. 85.