Translations

The previous discrete spectrum state vector formalism can be generalized also to continuos cases, in practice, by replacing

- summations with integrations
- Kronecker's δ-function with Dirac's δ-function.

A typical continuous case is the measurement of position:

- the operator x corresponding to the measurement of the x-coordinate of the position is Hermitean,
- the eigenvalues $\{x'\}$ of x are real,
- the eigenvectors $\{|x'\rangle\}$ form a complete basis.

So, we have

$$\begin{aligned} x|x'\rangle &= x'|x'\rangle \\ 1 &= \int_{-\infty}^{\infty} dx'\,|x'\rangle\langle x'| \\ |\alpha\rangle &= \int_{-\infty}^{\infty} dx'\,|x'\rangle\langle x'|\alpha\rangle, \end{aligned}$$

where $|\alpha\rangle$ is an arbitrary state vector. The quantity $\langle x'|\alpha\rangle$ is called a *wave function* and is usually written down using the function notation

$$\langle x'|\alpha\rangle = \psi_\alpha(x').$$

Obviously, looking at the expansion

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx'\,|x'\rangle\langle x'|\alpha\rangle,$$

the quantity $|\psi_\alpha(x')|^2 dx'$ can be interpreted according to the postulate 3 as the probability for the state being localized in the neighborhood $(x', x'+dx')$ of the point x'. The position can be generalized to three dimension. We denote by $|\boldsymbol{x}'\rangle$ the *simultaneous* eigenvector of the operators x, y and z, i.e.

$$\begin{aligned} |\boldsymbol{x}'\rangle &\equiv |x', y', z'\rangle \\ x|\boldsymbol{x}'\rangle &= x'|\boldsymbol{x}'\rangle, \quad y|\boldsymbol{x}'\rangle = y'|\boldsymbol{x}'\rangle, \quad z|\boldsymbol{x}'\rangle = z'|\boldsymbol{x}'\rangle. \end{aligned}$$

The exsistence of the common eigenvector requires commutativity of the corresponding operators:

$$[x_i, x_j] = 0.$$

Let us suppose that the state of the system is localized at the point $\boldsymbol{x}'$. We consider an operation which transforms this state to another state, this time localized at the point $\boldsymbol{x}' + d\boldsymbol{x}'$, all other observables keeping their values. This operation is called an *infinitesimal translation.* The corresponding operator is denoted by $\mathcal{T}(d\boldsymbol{x}')$:

$$\mathcal{T}(d\boldsymbol{x}')|\boldsymbol{x}'\rangle = |\boldsymbol{x}' + d\boldsymbol{x}'\rangle.$$

The state vector on the right hand side is again an eigenstate of the position operator. Quite obviously, the vector $|\boldsymbol{x}'\rangle$ is *not* an eigenstate of the operator $\mathcal{T}(d\boldsymbol{x}')$.

The effect of an infinitesimal translation on an arbitrary state can be seen by expanding it using position eigenstates:

$$\begin{aligned} |\alpha\rangle \longrightarrow \mathcal{T}(d\boldsymbol{x}'')|\alpha\rangle &= \mathcal{T}(d\boldsymbol{x}'')\int d^3x'\,|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|\alpha\rangle \\ &= \int d^3x'\,|\boldsymbol{x}' + d\boldsymbol{x}''\rangle\langle\boldsymbol{x}'|\alpha\rangle \\ &= \int d^3x'\,|\boldsymbol{x}'\rangle\langle\boldsymbol{x}' - d\boldsymbol{x}''|\alpha\rangle, \end{aligned}$$

because $\boldsymbol{x}'$ is an ordinary integration variable.
To construct $\mathcal{T}(d\boldsymbol{x}')$ explicitely we need extra constraints:

1. it is natural to require that it preserves the normalization (i.e. the conservation of probability) of the state vectors:

   $$\langle\alpha|\alpha\rangle = \langle\alpha|\mathcal{T}^\dagger(d\boldsymbol{x}')\mathcal{T}(d\boldsymbol{x}')|\alpha\rangle.$$

   This is satisfied if $\mathcal{T}(d\boldsymbol{x}')$is unitary, i.e.

   $$\mathcal{T}^\dagger(d\boldsymbol{x}')\mathcal{T}(d\boldsymbol{x}') = 1.$$

2. we require that two consecutive translations are equivalent to a single combined transformation:

   $$\mathcal{T}(d\boldsymbol{x}')\mathcal{T}(d\boldsymbol{x}'') = \mathcal{T}(d\boldsymbol{x}' + d\boldsymbol{x}'').$$

3. the translation to the opposite direction is equivalent to the inverse of the original translation:

   $$\mathcal{T}(-d\boldsymbol{x}') = \mathcal{T}^{-1}(d\boldsymbol{x}').$$

4. we end up with the identity operator when $d\boldsymbol{x}' \to 0$:

   $$\lim_{d\boldsymbol{x}'\to 0} \mathcal{T}(d\boldsymbol{x}') = 1.$$

It is easy to see that the operator

$$\mathcal{T}(d\boldsymbol{x}') = 1 - i\boldsymbol{K}\cdot d\boldsymbol{x}',$$

where the components K_x, K_y and K_z of the vector $\boldsymbol{K}$ are *Hermitean operators*, satisfies all four conditions.
Using the definition

$$\mathcal{T}(d\boldsymbol{x}')|\boldsymbol{x}'\rangle = |\boldsymbol{x}' + d\boldsymbol{x}'\rangle$$

we can show that

$$[\boldsymbol{x}, \mathcal{T}(d\boldsymbol{x}')] = d\boldsymbol{x}'.$$

Substituting the explicit reprersentation

$$\mathcal{T}(d\boldsymbol{x}') = 1 - i\boldsymbol{K}\cdot d\boldsymbol{x}'$$

it is now easy to prove the commutation relation

$$[x_i, K_j] = i\delta_{ij}.$$

The equations

$$\begin{aligned} \mathcal{T}(d\boldsymbol{x}') &= 1 - i\boldsymbol{K}\cdot d\boldsymbol{x}' \\ \mathcal{T}(d\boldsymbol{x}')|\boldsymbol{x}'\rangle &= |\boldsymbol{x}' + d\boldsymbol{x}'\rangle \end{aligned}$$

can be considered as the definition of $\boldsymbol{K}$.
One would expect the operator $\boldsymbol{K}$ to have something to do with the momentum. It is, however, not quite the momentum, because its dimension is 1/length. Writing

$$\boldsymbol{p} = \hbar \boldsymbol{K}$$

we get an operator $\boldsymbol{p}$, with dimension of momentum. We take this as the definition of the momemtum. The commutation relation

$$[x_i, K_j] = i\delta_{ij}$$

can now be written in a familiar form like

$$[x_i, p_j] = i\hbar\delta_{ij}.$$

Finite translations
Consider translation of the distance $\Delta x'$ along the x-axis:

$$\mathcal{T}(\Delta x' \hat{\boldsymbol{x}})|\boldsymbol{x}'\rangle = |\boldsymbol{x}' + \Delta x' \hat{\boldsymbol{x}}\rangle.$$

We construct this translation combining infinitesimal translations of distance $\Delta x'/N$ letting $N \to \infty$:

$$\begin{aligned} \mathcal{T}(\Delta x' \hat{\boldsymbol{x}}) &= \lim_{N\to\infty} \left(1 - \frac{ip_x \Delta x'}{N\hbar}\right)^N \\ &= \exp\left(-\frac{ip_x \Delta x'}{\hbar}\right). \end{aligned}$$

It is relatively easy to show that the translation operators satisfy

$$[\mathcal{T}(\Delta y' \hat{\boldsymbol{y}}), \mathcal{T}(\Delta x' \hat{\boldsymbol{x}})] = 0,$$

so it follows that

$$[p_y, p_x] = 0.$$

Generally

$$[p_i, p_j] = 0.$$

This commutation relation tells that it is possible to construct a state vector which is a simultaneous eigenvector of all components of the momentum operator, i.e. there exists a vector

$$|\boldsymbol{p}'\rangle \equiv |p'_x, p'_y, p'_z\rangle,$$

so that

$$p_x|\boldsymbol{p}'\rangle = p'_x|\boldsymbol{p}'\rangle, \quad p_y|\boldsymbol{p}'\rangle = p'_y|\boldsymbol{p}'\rangle, \quad p_z|\boldsymbol{p}'\rangle = p'_z|\boldsymbol{p}'\rangle.$$

The effect of the translation $\mathcal{T}(d\boldsymbol{x}')$ on an eigenstate of the momentum operator is

$$\mathcal{T}(d\boldsymbol{x}')|\boldsymbol{p}'\rangle = \left(1 - \frac{i\boldsymbol{p}\cdot d\boldsymbol{x}'}{\hbar}\right)|\boldsymbol{p}'\rangle = \left(1 - \frac{i\boldsymbol{p}'\cdot d\boldsymbol{x}'}{\hbar}\right)|\boldsymbol{p}'\rangle.$$

The state $|\boldsymbol{p}'\rangle$ is thus an eigenstate of $\mathcal{T}(d\boldsymbol{x}')$: a result, which we could have predicted because

$$[\boldsymbol{p}, \mathcal{T}(d\boldsymbol{x}')] = 0.$$

Note The eigenvalues of $\mathcal{T}(d\boldsymbol{x}')$ are complex because it is not Hermitean.

So, we have derived the *canonical commutation relations* or *fundamental commutation relations*

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij}.$$

Recall, that the projection of the state $|\alpha\rangle$ along the state vector $|x'\rangle$ was called the *wave function* and was denoted like $\psi_\alpha(x')$. Since the vectors $|x'\rangle$ form a complete basis the scalar product between the states $|\alpha\rangle$ and $|\beta\rangle$ can be written with the help of the wave functions as

$$\langle\beta|\alpha\rangle = \int dx'\, \langle\beta|x'\rangle\langle x'|\alpha\rangle = \int dx'\, \psi^*_\beta(x')\psi_\alpha(x'),$$

i.e. $\langle\beta|\alpha\rangle$ tells how much the wave functions overlap. If $|a'\rangle$ is an eigenstate of A we define the corresponding *eigenfunction* $u_{a'}(x')$ like

$$u_{a'}(x') = \langle x'|a'\rangle.$$

An arbitrary wave function $\psi_\alpha(x')$ can be expanded using eigenfunctions as

$$\psi_\alpha(x') = \sum_{a'} c_{a'} u_{a'}(x').$$

The matrix element $\langle\beta|A|\alpha\rangle$ of an operator A can also be expressed with the help of eigenfunctions like

$$\begin{aligned} \langle\beta|A|\alpha\rangle &= \int dx' \int dx''\, \langle\beta|x'\rangle\langle x'|A|x''\rangle\langle x''|\alpha\rangle \\ &= \int dx' \int dx''\, \psi^*_\beta(x')\langle x'|A|x''\rangle\psi_\alpha(x''). \end{aligned}$$

To apply this formula we have to evaluate the matrix elements $\langle x'|A|x''\rangle$, which in general are functions of the two variables x' and x''. When A depends only on the position operator x,

$$A = f(x),$$

the calculations are much simpler:

$$\langle\beta|f(x)|\alpha\rangle = \int dx'\, \psi^*_\beta(x')f(x')\psi_\alpha(x').$$

Note $f(x)$ on the left hand side is an operator while $f(x')$ on the right hand side is an ordinary number.

Momentum operator $\boldsymbol{p}$ in position basis $\{|\boldsymbol{x}'\rangle\}$
For simplicity we consider the one dimensional case. According to the equation

$$\begin{aligned} \mathcal{T}(d\boldsymbol{x}'')|\alpha\rangle &= \mathcal{T}(d\boldsymbol{x}'') \int d^3x'\, |\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|\alpha\rangle \\ &= \int d^3x'\, |\boldsymbol{x}' + d\boldsymbol{x}''\rangle\langle\boldsymbol{x}'|\alpha\rangle \\ &= \int d^3x'\, |\boldsymbol{x}'\rangle\langle\boldsymbol{x}' - d\boldsymbol{x}''|\alpha\rangle \end{aligned}$$

we can write

$$\left(1 - \frac{ip\,dx''}{\hbar}\right)|\alpha\rangle$$

$$
\begin{aligned}
&= \int dx'\, \mathcal{T}(dx'')|x'\rangle\langle x'|\alpha\rangle \\
&= \int dx'\, |x'\rangle\langle x' - dx''|\alpha\rangle \\
&= \int dx'\, |x'\rangle \left(\langle x'|\alpha\rangle - dx'' \frac{\partial}{\partial x'}\langle x'|\alpha\rangle \right).
\end{aligned}
$$

In the last step we have expanded $\langle x' - dx''|\alpha\rangle$ as Taylor series. Comparing both sides of the equation we see that

$$p|\alpha\rangle = \int dx'\, |x'\rangle \left(-i\hbar \frac{\partial}{\partial x'}\langle x'|\alpha\rangle \right),$$

or, taking scalar product with a position eigenstate on both sides,

$$\langle x'|p|\alpha\rangle = -i\hbar \frac{\partial}{\partial x'}\langle x'|\alpha\rangle.$$

In particular, if we choose $|\alpha\rangle = |x'\rangle$ we get

$$\langle x'|p|x''\rangle = -i\hbar \frac{\partial}{\partial x'}\delta(x' - x'').$$

Taking scalar product with an arbitrary state vector $|\beta\rangle$ on both sides of

$$p|\alpha\rangle = \int dx'\, |x'\rangle \left(-i\hbar \frac{\partial}{\partial x'}\langle x'|\alpha\rangle \right)$$

we get the important relation

$$\langle\beta|p|\alpha\rangle = \int dx' \psi_\beta^*(x') \left(-i\hbar \frac{\partial}{\partial x'} \right) \psi_\alpha(x').$$

Just like the position eigenvalues also the momentum eigenvalues p' comprise a continuum. Analogically we can define the *momentum space wave function* as

$$\langle p'|\alpha\rangle = \phi_\alpha(p').$$

We can move between the momentum and configuration space representations with help of the relations

$$
\begin{aligned}
\psi_\alpha(x') &= \langle x'|\alpha\rangle = \int dp'\, \langle x'|p'\rangle\langle p'|\alpha\rangle \\
\phi_\alpha(p') &= \langle p'|\alpha\rangle = \int dx'\, \langle p'|x'\rangle\langle x'|\alpha\rangle.
\end{aligned}
$$

The transformation function $\langle x'|p'\rangle$ can be evaluated by substituting a momentum eigenvector $|p'\rangle$ for $|\alpha\rangle$ into

$$\langle x'|p|\alpha\rangle = -i\hbar \frac{\partial}{\partial x'}\langle x'|\alpha\rangle.$$

Then

$$\langle x'|p|p'\rangle = p'\langle x'|p'\rangle = -i\hbar \frac{\partial}{\partial x'}\langle x'|p'\rangle.$$

The solution of this differential equation is

$$\langle x'|p'\rangle = C \exp\left(\frac{ip'x'}{\hbar} \right),$$

where the normalization factor C can be determined from the identity

$$\langle x'|x''\rangle = \int dp'\, \langle x'|p'\rangle\langle p'|x''\rangle.$$

Here the left hand side is simply $\delta(x' - x'')$ and the integration of the left hand side gives $2\pi\hbar|C|^2\delta(x' - x'')$. Thus the transformation function is

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ip'x'}{\hbar} \right),$$

and the relations

$$
\begin{aligned}
\psi_\alpha(x') &= \langle x'|\alpha\rangle = \int dp'\, \langle x'|p'\rangle\langle p'|\alpha\rangle \\
\phi_\alpha(p') &= \langle p'|\alpha\rangle = \int dx'\, \langle p'|x'\rangle\langle x'|\alpha\rangle.
\end{aligned}
$$

can be written as familiar Fourier transforms

$$
\begin{aligned}
\psi_\alpha(x') &= \left[\frac{1}{\sqrt{2\pi\hbar}} \right] \int dp'\, \exp\left(\frac{ip'x'}{\hbar} \right) \phi_\alpha(p') \\
\phi_\alpha(p') &= \left[\frac{1}{\sqrt{2\pi\hbar}} \right] \int dx'\, \exp\left(-\frac{ip'x'}{\hbar} \right) \psi_\alpha(x').
\end{aligned}
$$