Angular momentum

O(3)

We consider *active* rotations.
3×3 orthogonal matrix $R \iff$ rotation in$\mathcal{R}^3$.

Number of parameters

1. RR^T symmetric $\Rightarrow RR^T$ has 6 independent parameters $\Rightarrow$ orthogonality condition $RR^T = 1$ gives 6 independent equations $\Rightarrow R$ has $9 - 6 = 3$ free parameters.

2. Rotation around $\hat{\boldsymbol{n}}$ (2 angles) by the angle $\phi \Rightarrow 3$ parameters.

3. $\hat{\boldsymbol{n}}\phi$ vector $\Rightarrow$ 3 parameters.

3×3 orthogonal matrices form a group with respect to the matrix multiplication:

1. R_1R_2 is orthogonal if R_1 and R_2 are orthogonnal.

2. $R_1(R_2R_3) = (R_1R_2)R_3$, associativity.

3. $\exists$ identity $I =$ the unit matrix.

4. if R is orthogonal, then also the inverse matrix $R^{-1} = R^T$ is orthogonal.

The group is called O(3).
Generally rotations do not commute,

$$R_1R_2 \neq R_2R_1,$$

so the group is non-Abelian.
Rotations around a common axis commute.
Rotation around z-axis:

$$\begin{aligned} R_z(\phi) &= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ R_z \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \\ z \end{pmatrix}. \end{aligned}$$

Infinitesimal rotations up to the order $\mathcal{O}(\epsilon^2)$:

$$\begin{aligned} R_z(\epsilon) &= \begin{pmatrix} 1-\frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1-\frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ R_x(\epsilon) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ 0 & \epsilon & 1-\frac{\epsilon^2}{2} \end{pmatrix}, \\ R_y(\epsilon) &= \begin{pmatrix} 1-\frac{\epsilon^2}{2} & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\frac{\epsilon^2}{2} \end{pmatrix}. \end{aligned}$$

We see that

$$\begin{aligned} R_x(\epsilon)R_y(\epsilon) - R_y(\epsilon)R_x(\epsilon) &= \begin{pmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= R_z(\epsilon^2) - 1. \end{aligned}$$

In a Hilbert space we associate

$$R \longleftrightarrow \mathcal{D}(R),$$

i.e.

$$|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle.$$

We define the angular momentum (J) so that (we are not employing properties of the classical angular momentum $\boldsymbol{x} \times \boldsymbol{p}$)

$$\mathcal{D}(\hat{\boldsymbol{n}}, d\phi) = 1 - i\left(\frac{\boldsymbol{J}\cdot\hat{\boldsymbol{n}}}{\hbar}\right)d\phi$$

and require that the rotation operator $\mathcal{D}$

- is unitary,
- is decomposable,
- $\mathcal{D} \to 1$, when $d\phi \to 0$.

We see that $\boldsymbol{J}$ must be Hermitean, i.e.

$$\boldsymbol{J}^\dagger = \boldsymbol{J}.$$

Moreover, we require that $\mathcal{D}$ satisfies the same group properties as R, i.e.

$$\mathcal{D}_x(\epsilon)\mathcal{D}_y(\epsilon) - \mathcal{D}_y(\epsilon)\mathcal{D}_x(\epsilon) = \mathcal{D}_z(\epsilon^2) - 1.$$

Since rotations around a common axis commute a finite rotation can be constructed as

$$\begin{aligned} \mathcal{D}(\hat{\boldsymbol{n}}\phi) &= \lim_{N\to\infty}\left[1 - i\left(\frac{\boldsymbol{J}\cdot\hat{\boldsymbol{n}}}{\hbar}\right)\left(\frac{\phi}{N}\right)\right]^N \\ &= \exp\left(-\frac{i\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\phi}{\hbar}\right) \\ &= 1 - i\frac{\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\phi}{\hbar} - \frac{(\boldsymbol{J}\cdot\hat{\boldsymbol{n}})^2\phi^2}{2\hbar^2} + \cdots. \end{aligned}$$

We apply this up to the order $\mathcal{O}(\epsilon^2)$:

$$\begin{aligned} &\left(1 - \frac{iJ_x\epsilon}{\hbar} - \frac{J_x^2\epsilon^2}{2\hbar^2}\right)\left(1 - \frac{iJ_y\epsilon}{\hbar} - \frac{J_y^2\epsilon^2}{2\hbar^2}\right) \\ &- \left(1 - \frac{iJ_y\epsilon}{\hbar} - \frac{J_y^2\epsilon^2}{2\hbar^2}\right)\left(1 - \frac{iJ_x\epsilon}{\hbar} - \frac{J_x^2\epsilon^2}{2\hbar^2}\right) \\ &= -\frac{1}{\hbar^2}J_xJ_y\epsilon^2 + \frac{1}{\hbar^2}J_yJ_x + \mathcal{O}(\epsilon^3) \\ &= 1 - i\frac{J_z\epsilon^2}{\hbar} - 1. \end{aligned}$$

We see that

$$[J_x, J_y] = i\hbar J_z.$$

Similarly for other components:

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k.$$

We consider:

$$\langle J_x \rangle \equiv \langle \alpha | J_x | \alpha \rangle \longrightarrow$$
$$_R\langle \alpha | J_x | \alpha \rangle_R = \langle \alpha | \mathcal{D}_z^\dagger(\phi) J_x \mathcal{D}_z(\phi) | \alpha \rangle.$$

We evaluate

$$\mathcal{D}_z^\dagger(\phi) J_x \mathcal{D}_z(\phi) = \exp\left(\frac{iJ_z\phi}{\hbar}\right) J_x \exp\left(-\frac{iJ_z\phi}{\hbar}\right)$$

applying the Baker-Hausdorff lemma

$$e^{iG\lambda} A e^{-iG\lambda} = A + i\lambda[G,A] + \left(\frac{i^2\lambda^2}{2!}\right)[G,[G,A]] + \cdots + \left(\frac{i^n\lambda^n}{n!}\right)[G,[G,[G,\ldots[G,A]]]\ldots] + \cdots$$

where G is Hermitean. So we need the commutators

$$\begin{aligned} [J_z, J_x] &= i\hbar J_y \\ [J_z,[J_z,J_x]] &= i\hbar[J_z,J_y] = \hbar^2 J_x \\ [J_z,[J_z,[J_z,J_x]]] &= \hbar^2[J_z,J_x] = i\hbar^3 J_y \\ &\vdots \end{aligned}$$

Substituting into the Baker-Hausdorff lemma we get

$$\mathcal{D}_z^\dagger(\phi) J_x \mathcal{D}_z(\phi) = J_x \cos\phi - J_y \sin\phi.$$

Thus the expectation value is

$$\langle J_x \rangle \longrightarrow {}_R\langle \alpha | J_x | \alpha \rangle_R = \langle J_x \rangle \cos\phi - \langle J_y \rangle \sin\phi.$$

Correspondingly we get for the other components

$$\begin{aligned} \langle J_y \rangle &\longrightarrow \langle J_y \rangle \cos\phi + \langle J_x \rangle \sin\phi \\ \langle J_z \rangle &\longrightarrow \langle J_z \rangle. \end{aligned}$$

We see that the components of the expectation value of the angular momentum operator transform in rotations like a vector in $\mathcal{R}^3$:

$$\langle J_k \rangle \longrightarrow \sum_l R_{kl} \langle J_l \rangle.$$

Euler angles

1. Rotate the system counterclockwise by the angle α around the z-axis. The y-axis of of the system coordinates rotates then to a new position y'.

2. Rotate the system counterclockwise by the angle β around the y'-axis. The system z-axis rotates now to a new position z'.

3. Rotate the system counterclockwise by the angle γ around the z'-axis.

Using matrices:

$$R(\alpha,\beta,\gamma) \equiv R_{z'}(\gamma) R_{y'}(\beta) R_z(\alpha).$$

Now

$$\begin{aligned} R_{y'}(\beta) &= R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) \\ R_{z'}(\gamma) &= R_{y'}(\beta) R_z(\gamma) R_{y'}^{-1}(\beta), \end{aligned}$$

so

$$\begin{aligned} R(\alpha,\beta,\gamma) &= R_{y'}(\beta) R_z(\gamma) R_{y'}^{-1}(\beta) R_{y'}(\beta) R_z(\alpha) \\ &= R_{y'}(\beta) R_z(\alpha) R_z(\gamma) \\ &= R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) R_z(\alpha) R_z(\gamma) \\ &= R_z(\alpha) R_y(\beta) R_z(\gamma). \end{aligned}$$

Correspondingly

$$\mathcal{D}(\alpha,\beta,\gamma) = \mathcal{D}_z(\alpha) \mathcal{D}_y(\beta) \mathcal{D}_z(\gamma).$$