SU(2)

In the two dimensional space

$$\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$$

the spin operators

$$\begin{aligned}
S_x &= \left(\frac{\hbar}{2}\right)\{(|S_z;\uparrow\rangle\langle S_z;\downarrow|) + (|S_z;\downarrow\rangle\langle S_z;\uparrow|)\} \\
S_y &= \left(\frac{i\hbar}{2}\right)\{-(|S_z;\uparrow\rangle\langle S_z;\downarrow|) + (|S_z;\downarrow\rangle\langle S_z;\uparrow|)\} \\
S_z &= \left(\frac{\hbar}{2}\right)\{(|S_z;\uparrow\rangle\langle S_z;\uparrow|) - (|S_z;\downarrow\rangle\langle S_z;\downarrow|)\}
\end{aligned}$$

satisfy the angular momentum commutation relations

$$[S_x, S_y] = i\hbar S_z + \text{cyclic permutations.}$$

Thus the smallest dimension where these commutation relations can be realized is 2.
The state

$$|\alpha\rangle = |S_z;\uparrow\rangle\langle S_z;\uparrow|\alpha\rangle + |S_z;\downarrow\rangle\langle S_z;\downarrow|\alpha\rangle$$

behaves in the rotation

$$\mathcal{D}_z(\phi) = \exp\left(-\frac{iS_z\phi}{\hbar}\right)$$

like

$$\begin{aligned}
\mathcal{D}_z(\phi)|\alpha\rangle &= \exp\left(-\frac{iS_z\phi}{\hbar}\right)|\alpha\rangle \\
&= e^{-i\phi/2}|S_z;\uparrow\rangle\langle S_z;\uparrow|\alpha\rangle \\
&\quad + e^{i\phi/2}|S_z;\downarrow\rangle\langle S_z;\downarrow|\alpha\rangle.
\end{aligned}$$

In particular:

$$\mathcal{D}_z(2\pi)|\alpha\rangle = -|\alpha\rangle.$$

Spin precession
When the Hamiltonian is

$$H = \omega_c S_z$$

the time evolution operator is

$$\mathcal{U}(t,0) = \exp\left(-\frac{iS_z\omega_c t}{\hbar}\right) = \mathcal{D}_z(\omega_c t).$$

Looking at the equations

$$\begin{aligned}
\langle J_x\rangle &\longrightarrow_R \langle J_x\rangle\cos\phi - \langle J_y\rangle\sin\phi \\
\langle J_y\rangle &\longrightarrow_R \langle J_y\rangle\cos\phi + \langle J_x\rangle\sin\phi \\
\langle J_z\rangle &\longrightarrow_R \langle J_z\rangle
\end{aligned}$$

one can read that

$$\begin{aligned}
\langle S_x\rangle_t &= \langle S_x\rangle_{t=0}\cos\omega_c t - \langle S_y\rangle_{t=0}\sin\omega_c t \\
\langle S_y\rangle_t &= \langle S_y\rangle_{t=0}\cos\omega_c t + \langle S_x\rangle_{t=0}\sin\omega_c t \\
\langle S_z\rangle_t &= \langle S_z\rangle_{t=0}.
\end{aligned}$$

We see that

- the spin returns to its original direction after time $t = 2\pi/\omega_c$.
- the wave vector returns to its original value after time $t = 4\pi/\omega_c$.

Matrix representation
In the basis $\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$ the base vectors are represented as

$$\begin{aligned}
|S_z;\uparrow\rangle &\mapsto \begin{pmatrix}1\\0\end{pmatrix} \equiv \chi_\uparrow \qquad |S_z;\downarrow\rangle \mapsto \begin{pmatrix}0\\1\end{pmatrix} \equiv \chi_\downarrow \\
\langle S_z;\uparrow| &\mapsto (1,0) \equiv \chi_\uparrow^\dagger \qquad \langle S_z;\downarrow| \mapsto (0,1) \equiv \chi_\downarrow^\dagger,
\end{aligned}$$

so an arbitrary state vector is represented as

$$\begin{aligned}
|\alpha\rangle &\mapsto \begin{pmatrix}\langle S_z;\uparrow|\alpha\rangle\\ \langle S_z;\downarrow|\alpha\rangle\end{pmatrix} \\
\langle\alpha| &\mapsto (\langle\alpha|S_z;\uparrow\rangle, \langle\alpha|S_z;\downarrow\rangle).
\end{aligned}$$

The column vector

$$\chi = \begin{pmatrix}\langle S_z;\uparrow|\alpha\rangle\\ \langle S_z;\downarrow|\alpha\rangle\end{pmatrix} \equiv \begin{pmatrix}c_\uparrow\\ c_\downarrow\end{pmatrix}$$

is called the *two component spinor*

Pauli's spin matrices
Pauli's spin matrices σ_i are defined via the relations

$$(S_k)_{ij} \equiv \left(\frac{\hbar}{2}\right)(\sigma_k)_{ij},$$

where the matrix elements are evaluated in the basis $\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$.
For example

$$S_1 = S_x = \left(\frac{\hbar}{2}\right)\{(|S_z;\uparrow\rangle\langle S_z;\downarrow|) + (|S_z;\downarrow\rangle\langle S_z;\uparrow|)\},$$

so

$$\begin{aligned}
(S_1)_{11} = (S_1)_{22} &= 0 \\
(S_1)_{12} = (S_1)_{21} &= \frac{\hbar}{2},
\end{aligned}$$

or

$$(S_1) = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Thus we get

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The spin matrices satisfy the anticommutation relations

$$\{\sigma_i, \sigma_j\} \equiv \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$$

and the commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

Moreover, we see that

$$\begin{aligned}\sigma_i^\dagger &= \sigma_i,\\ \det(\sigma_i) &= -1,\\ \mathrm{tr}(\sigma_i) &= 0.\end{aligned}$$

Often the collective vector notation

$$\boldsymbol{\sigma} \equiv \sigma_1\hat{\boldsymbol{x}} + \sigma_2\hat{\boldsymbol{y}} + \sigma_3\hat{\boldsymbol{z}}.$$

is used for spin matrices. For example we get

$$\begin{aligned}\boldsymbol{\sigma}\cdot\boldsymbol{a} &\equiv \sum_k a_k\sigma_k\\ &= \begin{pmatrix} +a_3 & a_1 - ia_2\\ a_1 + ia_2 & -a_3\end{pmatrix}.\end{aligned}$$

and

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) &= \sum_{j,k}\sigma_j a_j\sigma_k b_k\\ &= \sum_{j,k}\frac{1}{2}\left(\{\sigma_j,\sigma_k\} + [\sigma_j,\sigma_k]\right)a_j b_k\\ &= \sum_{j,k}(\delta_{jk} + i\epsilon_{jki}\sigma_i)a_j b_k\\ &= \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b}).\end{aligned}$$

A special case of the latter formula is

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})^2 = |\boldsymbol{a}|^2.$$

Now

$$\mathcal{D}(\hat{\boldsymbol{n}},\phi) = \exp\left(-\frac{i\boldsymbol{S}\cdot\hat{\boldsymbol{n}}\phi}{\hbar}\right) \mapsto \exp\left(-\frac{i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi}{2}\right) =$$
$$\boldsymbol{1}\cos\left(\frac{\phi}{2}\right) - i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\sin\left(\frac{\phi}{2}\right) =$$
$$\begin{pmatrix}\cos\left(\frac{\phi}{2}\right) - in_z\sin\left(\frac{\phi}{2}\right) & (-in_x - n_y)\sin\left(\frac{\phi}{2}\right)\\ (-in_x + n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z\sin\left(\frac{\phi}{2}\right)\end{pmatrix}$$

and the spinors behave in rotations like

$$\chi \longrightarrow \exp\left(-\frac{i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi}{2}\right)\chi.$$

Note the notation $\boldsymbol{\sigma}$ *does not* mean that $\boldsymbol{\sigma}$ would behave in rotations like a vector, $\sigma_k \longrightarrow {}_R\sigma_k$. Instead we have

$$\chi^\dagger\sigma_k\chi \longrightarrow \sum_l R_{kl}\chi^\dagger\sigma_l\chi.$$

For all directions $\hat{\boldsymbol{n}}$ one has

$$\exp\left(-\frac{i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi}{2}\right)\Bigg|_{\phi=2\pi} = -\boldsymbol{1},\quad \text{for any } \hat{\boldsymbol{n}}.$$

Euler's angles
The spinor rotation matrices corresponding to rotations around z and y axes are

$$\begin{aligned}\mathcal{D}_z(\alpha) &\mapsto \begin{pmatrix} e^{-i\alpha/2} & 0\\ 0 & e^{i\alpha/2}\end{pmatrix}\\ \mathcal{D}_y(\beta) &\mapsto \begin{pmatrix}\cos\beta/2 & -\sin\beta/2\\ \sin\beta/2 & \cos\beta/2\end{pmatrix}.\end{aligned}$$

With the help of Euler's angles α, β and γ the rotation matrices can be written as

$$\mathcal{D}(\alpha,\beta,\gamma) \mapsto \mathcal{D}^{(\frac{1}{2})}(\alpha,\beta,\gamma) =$$
$$\begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) & -e^{-i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right)\\ e^{i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) & e^{i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right)\end{pmatrix}.$$

We seek for the eigenspinor of the matrix $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$:

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\chi = \chi.$$

Now

$$\hat{\boldsymbol{n}} = \begin{pmatrix}\sin\beta\cos\alpha\\ \sin\beta\sin\alpha\\ \cos\beta\end{pmatrix},$$

so

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} = \begin{pmatrix}\cos\beta & \sin\beta e^{-i\alpha}\\ \sin\beta e^{i\alpha} & -\cos\beta\end{pmatrix}.$$

The state where the spin is parallel to the unit vector $\hat{\boldsymbol{n}}$, is obviously invariant under rotations

$$\mathcal{D}_{\hat{\boldsymbol{n}}}(\phi) = e^{-i\boldsymbol{S}\cdot\hat{\boldsymbol{n}}/\hbar}$$

and thus an eigenstate of the operator $\boldsymbol{S}\cdot\hat{\boldsymbol{n}}$.
This kind of state can be obtained by rotating the state $|S_z;\uparrow\rangle$

1. angle β around y axis,
2. angle α around z axis,

i.e.

$$\begin{aligned}\boldsymbol{S}\cdot\hat{\boldsymbol{n}}|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};\uparrow\rangle &= \boldsymbol{S}\cdot\hat{\boldsymbol{n}}\mathcal{D}(\alpha,\beta,0)|S_z;\uparrow\rangle\\ &= \left(\frac{\hbar}{2}\right)\mathcal{D}(\alpha,\beta,0)|S_z;\uparrow\rangle\\ &= \left(\frac{\hbar}{2}\right)|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};\uparrow\rangle.\end{aligned}$$

Correspondingly for spinors the vector

$$\chi = \mathcal{D}^{(\frac{1}{2})}(\alpha,\beta,0)|S_z;\uparrow\rangle = \begin{pmatrix}\cos\left(\frac{\beta}{2}\right)e^{-i\alpha/2}\\ \sin\left(\frac{\beta}{2}\right)e^{i\alpha/2}\end{pmatrix}$$

is an eigenstate of the matrix $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$.

SU(2)
As a representation of rotations the 2×2-matrices

$$\mathcal{D}^{(\frac{1}{2})}(\hat{\boldsymbol{n}},\phi) = e^{-i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi/2}$$

form obviously a group. These matrices have two characteristic properties:

1. unitarity
$$\left(\mathcal{D}^{(\frac{1}{2})}\right)^{\dagger} = \left(\mathcal{D}^{(\frac{1}{2})}\right)^{-1},$$
2. unimodularity
$$\left|\mathcal{D}^{(\frac{1}{2})}\right| = 1.$$

A unitary unimodular matrix can be written as
$$U(a,b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}.$$
The unimodularity condition gives
$$1 = |U| = |a|^2 + |b|^2,$$
and we are left with 3 free parameters.
The unitarity condition is automatically satisfied because
$$\begin{aligned} U(a,b)^{\dagger} U(a,b) &= \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \\ &= \begin{pmatrix} |a|^2+|b|^2 & 0 \\ 0 & |a|^2+|b|^2 \end{pmatrix} = 1. \end{aligned}$$
Matrices $U(a,b)$ form a group since

- the matrix
$$U(a_1,b_1)U(a_2,b_2) = U(a_1a_2 - b_1b_2^*, a_1b_2 + a_2^*b_1)$$
is unimodular because
$$\begin{aligned} &|U(a_1a_2 - b_1b_2^*, a_1b_2 + a_2^*b_1)| = \\ &\quad |a_1a_2 - b_1b_2^*|^2 + |a_1b_2 + a_2^*b_1|^2 = 1, \end{aligned}$$
and thus also unitary.
- as a unitary matrix U has the inverse matrix:
$$U^{-1}(a,b) = U^{\dagger}(a,b) = U(a^*, -b).$$
- the unit matrix 1 is unitary and unimodular.

The group is called SU(2).
Comparing with the previous spinor representation
$$\mathcal{D}^{(\frac{1}{2})}(\hat{\boldsymbol{n}},\phi) = \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) - in_z\sin\left(\frac{\phi}{2}\right) & (-in_x - n_y)\sin\left(\frac{\phi}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z\sin\left(\frac{\phi}{2}\right) \end{pmatrix}$$
we see that
$$\begin{aligned} \mathrm{Re}(a) &= \cos\left(\frac{\phi}{2}\right) & \mathrm{Im}(a) &= -n_z\sin\left(\frac{\phi}{2}\right) \\ \mathrm{Re}(b) &= -n_y\sin\left(\frac{\phi}{2}\right) & \mathrm{Im}(b) &= -n_x\sin\left(\frac{\phi}{2}\right). \end{aligned}$$
The complex numbers a and b are known as *Cayley-Klein's parameters.*
Note O(3) and SU(2) are *not* isomorphic.
Example
In O(3): 2π- and 4π-rotations $\mapsto 1$
In SU(2): 2π-rotation $\mapsto -1$ and 4π-rotation $\mapsto 1$.
The operations $U(a,b)$ and $U(-a,-b)$ in SU(2) correspond to a single matrix of O(3). The map SU(2) $\mapsto$ O(3) is thus 2 to 1. The groups are, however, locally isomorphic.