Angular momentum algebra

It is easy to see that the operator

$$\boldsymbol{J}^2 = J_x J_x + J_y J_y + J_z J_z$$

commutes with the operators J_x, J_y and J_z,

$$[\boldsymbol{J}^2, J_i] = 0.$$

We choose the component J_z and denote the common eigenstate of the operators $\boldsymbol{J}^2$ and J_z by $|j,m\rangle$. We know (QM II) that

$$\begin{aligned}\boldsymbol{J}^2|j,m\rangle &= j(j+1)\hbar^2|j,m\rangle,\ j=0,\frac{1}{2},1,\frac{3}{2},\ldots \\ J_z|j,m\rangle &= m\hbar|j,m\rangle,\ m=-j,-j+1,\ldots,j-1,j.\end{aligned}$$

We define the ladder operators J_+ and J_-:

$$J_\pm \equiv J_x \pm iJ_y.$$

They satisfy the commutation relations

$$\begin{aligned}[J_+,J_-] &= 2\hbar J_z \\ [J_z,J_\pm] &= \pm\hbar J_\pm \\ \left[\boldsymbol{J}^2,J_\pm\right] &= 0.\end{aligned}$$

We see that

$$J_zJ_+|j,m\rangle = \hbar J_+J_z|j,m\rangle = (m+1)\hbar J_+|j,m\rangle$$

and

$$\boldsymbol{J}^2J_+|j,m\rangle = J_+\boldsymbol{J}^2|j,m\rangle = j(j+1)\hbar J_+|j,m\rangle,$$

so we must have

$$J_+|j,m\rangle = c_+|j,m+1\rangle$$

The factor c_+ can be deduced from the normalization condition

$$\langle j,m|j',m'\rangle = \delta_{jj'}\delta_{mm'}.$$

We end up with

$$J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\hbar|j,m\pm 1\rangle.$$

Matrix elements will be

$$\begin{aligned}\langle j',m'|\boldsymbol{J}^2|j,m\rangle &= j(j+1)\hbar^2\delta_{j'j}\delta_{m'm} \\ \langle j',m'|J_z|j,m\rangle &= m\hbar\delta_{j'j}\delta_{m'm} \\ \langle j',m'|J_\pm|j,m\rangle &= \sqrt{(j\mp m)(j\pm m+1)}\hbar\delta_{j'j}\delta_{m',m\pm 1}.\end{aligned}$$

We define *Wigner's function*:

$$\mathcal{D}^{(j)}_{m'm}(R) = \langle j,m'|\exp\left(-\frac{i\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\phi}{\hbar}\right)|j,m\rangle.$$

Since

$$[\boldsymbol{J}^2,\mathcal{D}(R)] = [\boldsymbol{J}^2,\exp\left(-\frac{i\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\phi}{\hbar}\right)] = 0,$$

we see that $\mathcal{D}(R)$ does not chance the j-quantum number, so it cannot have non zero matrix elements between states with different j values.

The matrix with matrix elements $\mathcal{D}^{(j)}_{m'm}(R)$ is the $(2j+1)$-dimensional irreducible representation of the rotation operator $\mathcal{D}(R)$.

The matrices $\mathcal{D}^{(j)}_{m'm}(R)$ form a group:

- The product of matrices belongs to the group:
$$\mathcal{D}^{(j)}_{m''m}(R_1R_2) = \sum_{m'}\mathcal{D}^{(j)}_{m''m'}(R_1)\mathcal{D}^{(j)}_{m'm}(R_2),$$
where R_1R_2 is the combined rotation of the rotations R_1 and R_2,
- the inverse operation belongs to the group:
$$\mathcal{D}^{(j)}_{m'm}(R^{-1}) = \mathcal{D}^{(j)^*}_{mm'}(R).$$

The state vectors $|j,m\rangle$ transform in rotations like

$$\begin{aligned}\mathcal{D}(R)|j,m\rangle &= \sum_{m'}|j,m'\rangle\langle j,m'|\mathcal{D}(R)|j,m\rangle \\ &= \sum_{m'}|j,m'\rangle\mathcal{D}^{(j)}_{m'm}(R).\end{aligned}$$

With the help of the Euler angles

$$\begin{aligned}&\mathcal{D}^{(j)}_{m'm}(R) = \\ &\quad\langle j,m'|\exp\left(-\frac{iJ_z\alpha}{\hbar}\right)\exp\left(-\frac{iJ_y\beta}{\hbar}\right)\exp\left(-\frac{iJ_z\gamma}{\hbar}\right)|j,m\rangle \\ &\quad= e^{-i(m'\alpha+m\gamma)}d^{(j)}_{m'm}(\beta),\end{aligned}$$

where

$$d^{(j)}_{m'm}(\beta) \equiv \langle j,m'|\exp\left(-\frac{iJ_y\beta}{\hbar}\right)|j,m\rangle.$$

Functions $d^{(j)}_{m'm}$ can be evaluated using *Wigner's formula*

$$\begin{aligned}&d^{(j)}_{m'm}(\beta) = \\ &\quad\sum_k(-1)^{k-m+m'} \\ &\quad\times\frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-k)!k!(j-k-m')!(k-m+m')!} \\ &\quad\times\left(\cos\frac{\beta}{2}\right)^{2j-2k+m-m'}\times\left(\sin\frac{\beta}{2}\right)^{2k-m+m'}.\end{aligned}$$

Orbital angular momentum

The components of the classically analogous operator $\boldsymbol{L} = \boldsymbol{x}\times\boldsymbol{p}$ satisfy the commutation relations

$$[L_i,L_j] = i\epsilon_{ijk}\hbar L_k.$$

Using the spherical coordinates to label the position eigenstates,

$$|\boldsymbol{x}'\rangle = |r,\theta,\phi\rangle,$$

one can show that

$$\begin{aligned}\langle\boldsymbol{x}'|L_z|\alpha\rangle &= -i\hbar\frac{\partial}{\partial\phi}\langle\boldsymbol{x}'|\alpha\rangle \\ \langle\boldsymbol{x}'|L_x|\alpha\rangle &= -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta}-\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\langle\boldsymbol{x}'|\alpha\rangle \\ \langle\boldsymbol{x}'|L_y|\alpha\rangle &= -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\langle\boldsymbol{x}'|\alpha\rangle \\ \langle\boldsymbol{x}'|L_\pm|\alpha\rangle &= -i\hbar e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\phi}\right)\langle\boldsymbol{x}'|\alpha\rangle \\ \langle\boldsymbol{x}'|\boldsymbol{L}^2|\alpha\rangle &= -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] \\ &\quad\times\langle\boldsymbol{x}'|\alpha\rangle.\end{aligned}$$

We denote the common eigenstate of the operators $\boldsymbol{L}^2$ and L_z by the ket-vector $|l,m\rangle$, i.e.

$$\begin{aligned} L_z|l,m\rangle &= m\hbar|l,m\rangle \\ \boldsymbol{L}^2|l,m\rangle &= l(l+1)\hbar^2|l,m\rangle. \end{aligned}$$

Since $\mathcal{R}^3$ can be represented as the direct product

$$\mathcal{R}^3 = \mathcal{R} \times \Omega,$$

where Ω is the surface of the unit sphere (position=distance from the origin and direction) the position eigenstates can be written correspondingly as

$$|\boldsymbol{x}'\rangle = |r\rangle|\hat{\boldsymbol{n}}\rangle.$$

Here the state vectors $|\hat{\boldsymbol{n}}\rangle$ form a complete basis on the surface of the sphere, i.e.

$$\int d\Omega_{\hat{\boldsymbol{n}}}\, |\hat{\boldsymbol{n}}\rangle\langle\hat{\boldsymbol{n}}| = 1.$$

We define the spherical harmonic function:

$$Y_l^m(\theta,\phi) = Y_l^m(\hat{\boldsymbol{n}}) = \langle\hat{\boldsymbol{n}}|l,m\rangle.$$

The scalar product of the vector $\langle\hat{\boldsymbol{n}}|$ with the equations

$$\begin{aligned} L_z|l,m\rangle &= m\hbar|l,m\rangle \\ \boldsymbol{L}^2|l,m\rangle &= l(l+1)\hbar^2|l,m\rangle \end{aligned}$$

gives

$$-i\hbar\frac{\partial}{\partial\phi}Y_l^m(\theta,\phi) = m\hbar Y_l^m(\theta,\phi)$$

and

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right]Y_l^m = 0.$$

Y_l^m and $\mathcal{D}^{(l)}$

The state

$$|\hat{\boldsymbol{n}}\rangle = |\theta,\phi\rangle$$

is obtained from the state $|\hat{\boldsymbol{z}}\rangle$ rotating it first by the angle θ around y-axis and then by the angle ϕ around z-axis:

$$\begin{aligned} |\hat{\boldsymbol{n}}\rangle &= \mathcal{D}(R)|\hat{\boldsymbol{z}}\rangle \\ &= \mathcal{D}(\alpha=\phi,\beta=\theta,\gamma=0)|\hat{\boldsymbol{z}}\rangle \\ &= \sum_{l,m}\mathcal{D}(\phi,\theta,0)|l,m\rangle\langle l,m|\hat{\boldsymbol{z}}\rangle. \end{aligned}$$

Furthermore

$$\langle l,m|\hat{\boldsymbol{n}}\rangle = Y_l^{m*}(\theta,\phi) = \sum_m \mathcal{D}^{(l)}_{m'm}(\phi,\theta,0)\langle l,m|\hat{\boldsymbol{z}}\rangle.$$

Now

$$\langle l,m|\hat{\boldsymbol{z}}\rangle = Y_l^{m*}(0,\phi) = \sqrt{\frac{(2l+1)}{4\pi}}\delta_{m0},$$

so

$$Y_l^{m*}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}}\mathcal{D}^{(l)}_{m0}(\phi,\theta,\gamma=0)$$

or

$$\mathcal{D}^{(l)}_{m0}(\alpha,\beta,0) = \sqrt{\frac{4\pi}{(2l+1)}}Y_l^{m*}(\theta,\phi)\bigg|_{\beta,\alpha}.$$

As a special case

$$\mathcal{D}^{(l)}_{00}(\theta,\phi,0) = d^{(l)}_{00}(\theta) = P_l(\cos\theta).$$

Coupling of angular momenta

We consider two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. If now A_i is an operator in the space $\mathcal{H}_i$, the notation $A_1 \otimes A_2$ means the operator

$$A_1 \otimes A_2|\alpha\rangle_1 \otimes |\beta\rangle_2 = (A_1|\alpha\rangle_1) \otimes (A_2|\beta\rangle_2)$$

in the product space. Here $|\alpha\rangle_i \in \mathcal{H}_i$. In particular,

$$A_1 \otimes 1_2|\alpha\rangle_1 \otimes |\beta\rangle_2 = (A_1|\alpha\rangle_1) \otimes |\beta\rangle_2,$$

where 1_i is the identity operator of the space $\mathcal{H}_i$. Correspondingly $1_1 \otimes A_2$ operates only in the subspace $\mathcal{H}_2$ of the product space. Usually the subspace of the identity operators, or even the identity operator itself, is not shown, for example

$$A_1 \otimes 1_2 = A_1 \otimes 1 = A_1.$$

It is easy to verify that operators operating in different subspace commute, i.e.

$$[A_1 \otimes 1_2, 1_1 \otimes A_2] = [A_1, A_2] = 0.$$

In particular we consider two angular momenta $\boldsymbol{J_1}$ and $\boldsymbol{J_2}$ operating in two different Hilbert spaces. They commute:

$$[J_{1i}, J_{2j}] = 0.$$

The infinitesimal rotation affecting both Hilbert spaces is

$$\left(1 - \frac{i\boldsymbol{J}_1\cdot\hat{\boldsymbol{n}}\delta\phi}{\hbar}\right) \otimes \left(1 - \frac{i\boldsymbol{J}_2\cdot\hat{\boldsymbol{n}}\delta\phi}{\hbar}\right) = 1 - \frac{i(\boldsymbol{J}_1\otimes 1 + 1\otimes\boldsymbol{J}_2)\cdot\hat{\boldsymbol{n}}\delta\phi}{\hbar}.$$

The components of the total angular momentum

$$\boldsymbol{J} = \boldsymbol{J}_1 \otimes 1 + 1 \otimes \boldsymbol{J}_2 = \boldsymbol{J}_1 + \boldsymbol{J}_2$$

obey the commutation relations

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k,$$

i.e. $\boldsymbol{J}$ *is angular momentum.*

A finite rotation is constructed analogously:

$$\mathcal{D}_1(R) \otimes \mathcal{D}_2(R) = \exp\left(-\frac{\boldsymbol{J}_1\cdot\hat{\boldsymbol{n}}\phi}{\hbar}\right) \otimes \exp\left(-\frac{\boldsymbol{J}_2\cdot\hat{\boldsymbol{n}}\phi}{\hbar}\right).$$

Base vectors of the whole system

We seek in the product space $\{|j_1m_1\rangle \otimes |j_2m_2\rangle\}$ for the maximal set of commuting operators.

(i) $\boldsymbol{J}_1^2$, $\boldsymbol{J}_2^2$, J_{1z} and J_{2z}.

Their common eigenstates are simply direct products

$$|j_1j_2;m_1m_2\rangle \equiv |j_1,m_1\rangle\otimes|j_2,m_2\rangle.$$

If j_1 and j_2 can be deduced from the context we often denote

$$|m_1m_2\rangle = |j_1j_2;m_1m_2\rangle.$$

The quantum numbers are obtained from the (eigen)equations

$$\begin{aligned}
\boldsymbol{J}_1^2|j_1j_2;m_1m_2\rangle &= j_1(j_1+1)\hbar^2|j_1j_2;m_1m_2\rangle\\
J_{1z}|j_1j_2;m_1m_2\rangle &= m_1\hbar|j_1j_2;m_1m_2\rangle\\
\boldsymbol{J}_2^2|j_1j_2;m_1m_2\rangle &= j_2(j_2+1)\hbar^2|j_1j_2;m_1m_2\rangle\\
J_{2z}|j_1j_2;m_1m_2\rangle &= m_2\hbar|j_1j_2;m_1m_2\rangle.
\end{aligned}$$

(ii) $\boldsymbol{J}^2$, $\boldsymbol{J}_1^2$, $\boldsymbol{J}_2^2$ and J_z.
Their common eigenstate is denoted as

$$|j_1j_2;jm\rangle$$

or shortly

$$|jm\rangle = |j_1j_2;jm\rangle$$

if the quantum numbers j_1 and j_2 can be deduced from the context. The quantum numbers are obtained from the equations

$$\begin{aligned}
\boldsymbol{J}_1^2|j_1j_2;jm\rangle &= j_1(j_1+1)\hbar^2|j_1j_2;jm\rangle\\
\boldsymbol{J}_2^2|j_1j_2;jm\rangle &= j_2(j_2+1)\hbar^2|j_1j_2;jm\rangle\\
\boldsymbol{J}^2|j_1j_2;jm\rangle &= j(j+1)\hbar^2|j_1j_2;jm\rangle\\
J_z|j_1j_2;jm\rangle &= m\hbar|j_1j_2;jm\rangle.
\end{aligned}$$

Now

$$[\boldsymbol{J}^2,J_{1z}]\neq 0,\quad [\boldsymbol{J}^2,J_{2z}]\neq 0,$$

so we cannot add to the set (i) the operator $\boldsymbol{J}^2$, nor to the set (ii) the operators J_{1z} or J_{2z}. Both sets are thus maximal and the corresponding bases complete (and orthonormal), i.e.

$$\begin{aligned}
\sum_{j_1j_2}\sum_{m_1m_2}|j_1j_2;m_1m_2\rangle\langle j_1j_2;m_1m_2| &= 1\\
\sum_{j_1j_2}\sum_{jm}|j_1j_2;jm\rangle\langle j_1j_2;jm| &= 1.
\end{aligned}$$

In the subspace where the quantum numbers j_1 and j_2 are fixed we have the completeness relations

$$\begin{aligned}
\sum_{m_1m_2}|j_1j_2;m_1m_2\rangle\langle j_1j_2;m_1m_2| &= 1\\
\sum_{jm}|j_1j_2;jm\rangle\langle j_1j_2;jm| &= 1.
\end{aligned}$$

One can go from the basis (i) to the basis (ii) via the unitary transformation

$$|j_1j_2;jm\rangle = \sum_{m_1m_2}|j_1j_2;m_1m_2\rangle\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle,$$

so also the transformation matrix

$$(C)_{jm,m_1m_2} = \langle j_1j_2;m_1m_2|j_1j_2;jm\rangle$$

is unitary. The elements $\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle$ of the transformation matrix are called *Clebsch-Gordan's coefficients.*
Since

$$J_z = J_{1z}+J_{2z},$$

we must have

$$m = m_1+m_2,$$

so the Clebsch-Gordan coefficients satisfy the condition

$$\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle = 0,\ \text{if } m\neq m_1+m_2.$$

Further, we must have (QM II)

$$|j_1-j_2|\leq j\leq j_1+j_2.$$

It turns out, that the C-G coefficients can be chosen to be real, so the transformation matrix C is in fact orthogonal:

$$\begin{aligned}
&\sum_{jm}\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle\langle j_1j_2;m_1'm_2'|j_1j_2;jm\rangle\\
&\qquad = \delta_{m_1m_1'}\delta_{m_2m_2'}\\
&\sum_{m_1m_2}\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle\langle j_1j_2;m_1m_2|j_1j_2;j'm'\rangle\\
&\qquad = \delta_{jj'}\delta_{mm'}.
\end{aligned}$$

As a special case ($j'=j$ and $m'=m=m_1+m_2$) we get the normalization condition

$$\sum_{m_1m_2}|\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle|^2 = 1.$$

Recursion formulas
Operating with the ladder operators to the state $|j_1j_2;jm\rangle$ we get

$$\begin{aligned}
&J_\pm|j_1j_2;jm\rangle =\\
&\qquad (J_{1\pm}+J_{2\pm})\sum_{m_1m_2}|j_1j_2;m_1m_2\rangle\\
&\qquad\qquad \times\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle,
\end{aligned}$$

or

$$\begin{aligned}
&\sqrt{(j\mp m)(j\pm m+1)}|j_1j_2;j,m\pm 1\rangle\\
&\quad = \sum_{m_1'}\sum_{m_2'}\Big(\sqrt{(j_1\mp m_1')(j_1\pm m_1'+1)}\\
&\qquad\quad \times|j_1j_2;m_1'\pm 1,m_2'\rangle\\
&\qquad + \sqrt{(j_2\pm m_2')(j_2\pm m_2'+1)}\\
&\qquad\quad \times|j_1j_2;m_1',m_2'\pm 1\rangle\Big)\\
&\qquad \times\langle j_1j_2;m_1'm_2'|j_1j_2;jm\rangle.
\end{aligned}$$

Taking the scalar product on the both sides with the vector $\langle j_1j_2;m_1m_2|$ we get

$$\begin{aligned}
&\sqrt{(j\mp m)(j\pm m+1)}\langle j_1j_2;m_1m_2|j_1j_2;j,m\pm 1\rangle\\
&\quad = \sqrt{(j_1\mp m_1+1)(j_1\pm m_1)}\\
&\qquad \times\langle j_1j_2;m_1\mp 1,m_2|j_1j_2;jm\rangle\\
&\quad + \sqrt{(j_2\mp m_2+1)(j_2\pm m_2)}\\
&\qquad \times\langle j_1j_2;m_1,m_2\mp 1|j_1j_2;jm\rangle.
\end{aligned}$$

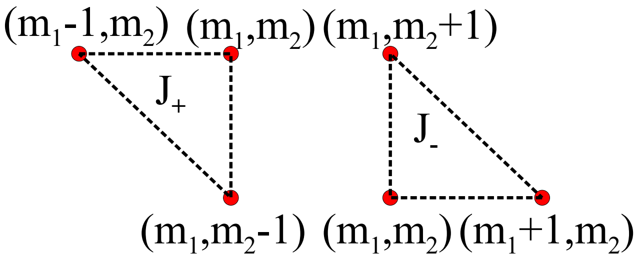
The Clebsch-Gordan coefficients are determined uniquely by

1. the recursion formulas.
2. the normalization condition
$$\sum_{m_1 m_2} |\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle|^2 = 1.$$
3. the sign conventions, for example
$$\langle j_1 j_2; j'm' | J_{1z} | j_1 j_2; jm\rangle \geq 0.$$

Note Due to the sign conventions the order of the coupling is essential:

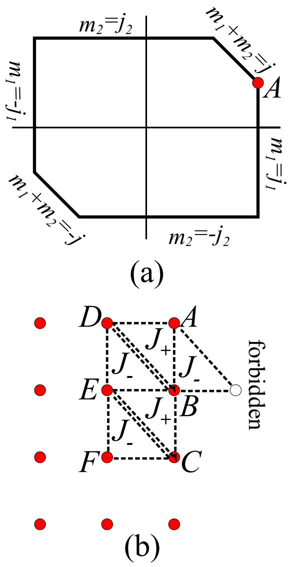
$$|j_1 j_2; jm\rangle = \pm |j_2 j_1; jm\rangle.$$

Graphical representation of recursion formulas



Recursion formula in $m_1 m_2$-plane
We fix j_1, j_2 and j. Then

$$|m_1| \leq j_1, \quad |m_2| \leq j_2, \quad |m_1 + m_2| \leq j.$$

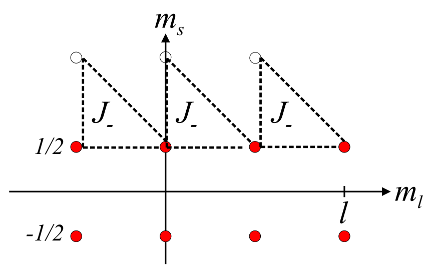


Using recursion formulas
We see that

1. every C-G coefficient depends on A,
2. the normalization condition determines the absolute value of A,
3. the sign is obtained from the sign conventions.

Example $\boldsymbol{L} + \boldsymbol{S}$-coupling.
Now

$$\begin{aligned} j_1 &= l = 0, 1, 2, \ldots \\ m_1 &= m_l = -l, -l+1, \ldots, l-1, l \\ j_2 &= s = \frac{1}{2} \\ m_2 &= m_s = \pm\frac{1}{2} \\ j &= \begin{cases} l \pm \frac{1}{2}, & \text{when } l > 0 \\ \frac{1}{2}, & \text{when } l = 0. \end{cases} \end{aligned}$$



Recursion when $j_1 = l$ and $j_2 = s = 1/2$
Using the selection rule

$$m_1 = m_l = m - \frac{1}{2}, \quad m_2 = m_s = \frac{1}{2}$$

and the shorthand notation the J_--recursion gives

$$\begin{aligned} &\sqrt{(l+\tfrac{1}{2}+m+1)(l+\tfrac{1}{2}-m)}\langle m-\tfrac{1}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m\rangle \\ &\quad = \sqrt{(l+m+\tfrac{1}{2})(l-m+\tfrac{1}{2})} \\ &\quad\quad \times\langle m+\tfrac{1}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m+1\rangle, \end{aligned}$$

or

$$\langle m-\tfrac{1}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m\rangle = \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}}\langle m+\tfrac{1}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m+1\rangle.$$

Applying the same recursion repeatedly we have

$$\begin{aligned} &\langle m-\tfrac{1}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m\rangle \\ &\quad = \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}}\sqrt{\frac{l+m+\frac{3}{2}}{l+m+\frac{5}{2}}}\langle m+\tfrac{3}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m+2\rangle \\ &\quad = \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}}\sqrt{\frac{l+m+\frac{3}{2}}{l+m+\frac{5}{2}}}\sqrt{\frac{l+m+\frac{5}{2}}{l+m+\frac{7}{2}}} \\ &\quad\quad \langle m+\tfrac{5}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m+3\rangle \\ &\quad = \\ &\quad\quad \vdots \\ &\quad = \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}\langle l,\tfrac{1}{2}|l+\tfrac{1}{2},l+\tfrac{1}{2}\rangle. \end{aligned}$$

If $j = j_{\max} = j_1 + j_2$ and $m = m_{\max} = j_1 + j_2$ one must have

$$\begin{aligned} &|j_1 j_2; jm\rangle = \\ &\quad \langle j_1 j_2; m_1 = j_1, m_2 = j_2 | j_1 j_2; jm\rangle |j_1 m_1\rangle |j_2 m_2\rangle. \end{aligned}$$

Now the normalization condition

$$|\langle j_1 j_2; m_1 = j_1, m_2 = j_2 | j_1 j_2; jm\rangle|^2 = 1$$

and the sign convention give

$$\langle j_1 j_2; m_1 = j_1, m_2 = j_2 | j_1 j_2; jm\rangle = 1.$$

Thus, in the case of the spin-orbit coupling,

$$\langle l, \tfrac{1}{2} | l + \tfrac{1}{2}, l + \tfrac{1}{2}\rangle = 1,$$

or

$$\langle m - \tfrac{1}{2}, \tfrac{1}{2} | l + \tfrac{1}{2}, m\rangle = \sqrt{\frac{l + m + \frac{1}{2}}{2l + 1}}.$$

With the help of the recursion relations, normalization condition and sign convention the rest of the C-G coefficients can be evaluated, too. We get

$$\begin{pmatrix} |j = l + \frac{1}{2}, m\rangle \\ |j = l - \frac{1}{2}, m\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{l + m + \frac{1}{2}}{2l + 1}} & \sqrt{\frac{l - m + \frac{1}{2}}{2l + 1}} \\ -\sqrt{\frac{l - m + \frac{1}{2}}{2l + 1}} & \sqrt{\frac{l + m + \frac{1}{2}}{2l + 1}} \end{pmatrix} \begin{pmatrix} |m_l = m - \frac{1}{2}, m_s = \frac{1}{2}\rangle \\ |m_l = m + \frac{1}{2}, m_s = -\frac{1}{2}\rangle \end{pmatrix}.$$

Rotation matrices
If $\mathcal{D}^{(j_1)}(R)$ is a rotation matrix in the base $\{|j_1 m_1\rangle | m_1 = -j_1, \ldots, j_1\}$ and $\mathcal{D}^{(j_2)}(R)$ a rotation matrix in the base $\{|j_2 m_2\rangle | m_2 = -j_2, \ldots, j_2\}$, then $\mathcal{D}^{(j_1)}(R) \otimes \mathcal{D}^{(j_2)}(R)$ is a rotation matrix in the $(2j_1 + 1) \times (2j_2 + 1)$-dimensional base $\{|j_1, m_1\rangle \otimes |j_2, m_2\rangle\}$. Selecting suitable superpositions of the vectors $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ the matrix takes the form like

$$\mathcal{D}^{(j_1)}(R) \otimes \mathcal{D}^{(j_2)}(R) \longrightarrow \begin{pmatrix} \boxed{\mathcal{D}^{(j_1+j_2)}} & & & 0 \\ & \boxed{\mathcal{D}^{(j_1+j_2-1)}} & & \\ & & \ddots & \\ 0 & & & \boxed{\mathcal{D}^{(|j_1-j_2|)}} \end{pmatrix}.$$

One can thus write

$$\mathcal{D}^{(j_1)} \otimes \mathcal{D}^{(j_2)} = \mathcal{D}^{(j_1+j_2)} \oplus \mathcal{D}^{(j_1+j_2-1)} \oplus \cdots \oplus \mathcal{D}^{(|j_1-j_2|)}.$$

As a check we can calculate the dimensions:

$$\begin{aligned}(2j_1 + 1)(2j_2 + 1) = {} \\ & 2(j_1 + j_2) + 1 + 2(j_1 + j_2 - 1) + 1 \\ & + \cdots + 2|j_1 - j_2| + 1.\end{aligned}$$

The matrix elements of the rotation operator satisfy

$$\begin{aligned}&\langle j_1 j_2; m_1 m_2 | \mathcal{D}(R) | j_1 j_2; m_1' m_2'\rangle \\ &\quad = \langle j_1 m_1 | \mathcal{D}(R) | j_1 m_1'\rangle \langle j_2 m_2 | \mathcal{D}(R) | j_2 m_2'\rangle \\ &\quad = \mathcal{D}^{(j_1)}_{m_1 m_1'}(R) \mathcal{D}^{(j_2)}_{m_2 m_2'}(R).\end{aligned}$$

On the other hand we have

$$\begin{aligned}&\langle j_1 j_2; m_1 m_2 | \mathcal{D}(R) | j_1 j_2; m_1' m_2'\rangle \\ &\quad = \sum_{jm} \sum_{j'm'} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \\ &\qquad \times \langle j_1 j_2; jm | \mathcal{D}(R) | j_1 j_2; j'm'\rangle \\ &\qquad \times \langle j_1 j_2; j'm' | j_1 j_2; m_1' m_2'\rangle \\ &\quad = \sum_{jm} \sum_{j'm'} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \mathcal{D}^{(j)}_{mm'}(R) \delta_{jj'} \\ &\qquad \times \langle j_1 j_2; m_1' m_2' | j_1 j_2; j'm'\rangle.\end{aligned}$$

We end up with the *Clebsch-Gordan series*

$$\begin{aligned}&\mathcal{D}^{(j_1)}_{m_1 m_1'}(R) \mathcal{D}^{(j_2)}_{m_2 m_2'}(R) = \\ &\quad \sum_j \sum_m \sum_{m'} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \\ &\quad \times \langle j_1 j_2; m_1' m_2' | j_1 j_2; jm'\rangle \mathcal{D}^{(j)}_{mm'}(R).\end{aligned}$$

As an application we have

$$\begin{aligned}&\int d\Omega Y_l^{m*}(\theta, \phi) Y_{l_1}^{m_1}(\theta, \phi) Y_{l_2}^{m_2}(\theta, \phi) \\ &\quad = \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)}{4\pi(2l + 1)}} \\ &\qquad \times \langle l_1 l_2; 00 | l_1 l_2; l0\rangle \langle l_1 l_2; m_1 m_2 | l_1 l_2; lm\rangle.\end{aligned}$$