Symmetry

Symmetries, constants of motion and degeneracies

Looking at the Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

of classical mechanics one can see that if the Lagrangian $\mathcal{L}(q_i, \dot{q}_i)$ is invariant under translations, i.e.

$$\mathcal{L}(q_i, \dot{q}_i) \longrightarrow \mathcal{L}(q_i + \delta q_i, \dot{q}_i) = \mathcal{L}(q_i, \dot{q}_i),$$

the momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

is a conserved quantity, i.e.

$$\frac{dp_i}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0.$$

Formulating classical mechanics using the Hamiltonian function $\mathcal{H}(q_i, p_i)$ the equations of motion take the forms

$$\begin{aligned} \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} \\ \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i}. \end{aligned}$$

Also looking at these one can see that if $\mathcal{H}$ is symmetric under the operation

$$q_i \longrightarrow q_i + \delta q_i$$

there exists a conserved quantity:

$$\dot{p}_i = 0.$$

In quantum mechanics operations of that kind (translations, rotations, ...) are associated with a unitary *symmetry operator*.
Let $\mathcal{S}$ be an arbitrary symmetry operator. We say that the Hamiltonian H is *symmetric*, if

$$[\mathcal{S}, H] = 0,$$

or due to the unitarity of the operator $\mathcal{S}$ equivalently

$$\mathcal{S}^\dagger H \mathcal{S} = H.$$

The matrix elements of the Hamiltonian are then invariant under that operation.
In the case of a continuum symmetry we can look at infinitesimal operations

$$\mathcal{S} = 1 - \frac{i\epsilon}{\hbar} G,$$

where the Hermitean operator G is the *generator* of that symmetry. From the condition

$$\mathcal{S}^\dagger H \mathcal{S} = H$$

it follows now

$$[G, H] = 0,$$

so according to the Heisenberg equation of motion

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$$

we have

$$\frac{dG}{dt} = 0.$$

In the Heisenberg formalism the observable G is thus a constant of motion. if H is invariant for example under

- translations then the momentum is constant of motion.
- rotations then the angular momentum is a constant motion.

Let us suppose now that the Hamiltonian is symmetric under the operations $\mathcal{S}$ generated by G:

$$\begin{aligned} \mathcal{S}^\dagger H \mathcal{S} &= H \\ [\mathcal{S}, H] &= 0 \\ [G, H] &= 0. \end{aligned}$$

Let $|g'\rangle$ be the eigenstates of G, i.e.

$$G|g'\rangle = g'|g'\rangle$$

and let the system at the moment t_0 be in the eigenstate $|g'\rangle$ of G. Since the time evolution operator is a functional of the Hamiltonian only,

$$U = U[H],$$

so

$$[G, U] = 0.$$

At the moment t we then have

$$\begin{aligned} G|g', t_0; t\rangle &= GU(t_0, t)|g'\rangle = U(t_0, t)G|g'\rangle \\ &= g'|g', t_0; t\rangle, \end{aligned}$$

or an eigenstate associated with a particular eigenvalue of G remains always an eigenstate belonging to the same eigenvalue.
Let us consider now the energy eigenstates $|n\rangle$, i.e.

$$H|n\rangle = E_n|n\rangle.$$

When the Hamiltonian is symmetric under the operations $\mathcal{S}$ we have

$$H(\mathcal{S}|n\rangle = \mathcal{S}H|n\rangle = E_n\mathcal{S}|n\rangle.$$

If now

$$|n\rangle \neq \mathcal{S}|n\rangle,$$

then the energy states E_n are degenerate. Thus a symmetry is also usually associated with a degeneracy.
Let us suppose now that the symmetry operation $\mathcal{S}$ can be parametrized with a continuous quantity, say λ:

$$\mathcal{S} = \mathcal{S}(\lambda).$$

When the Hamiltonian is symmetric under these operations all states $\mathcal{S}(\lambda)|n\rangle$ have the same energy.
Example Rotations $\mathcal{D}(R)$.
If

$$[\mathcal{D}(R), H] = 0,$$

then

$$[\boldsymbol{J}, H] = 0, \quad [\boldsymbol{J}^2, H] = 0.$$

So there exist simultaneous eigenvectors $|n; jm\rangle$ of the operators H, $\boldsymbol{J}^2$ ja J_z. Now all rotated states

$$\mathcal{D}(R)|n; jm\rangle$$

belong to the same energy eigenvalue. We know that

$$\mathcal{D}(R)|n; jm\rangle = \sum_{m'} |n; jm'\rangle \mathcal{D}^{(j)}_{m'm}(R),$$

that is, every rotated state is a superposition of $(2j+1)$ linearly independent states. The degeneracy is thus $(2j+1)$-fold.
Example Atomic electron.
The potential acting on an electron is of form

$$U = V(r) + V_{LS}\boldsymbol{L} \cdot \boldsymbol{S}.$$

Now

$$[\boldsymbol{J}, H] = 0, \quad [\boldsymbol{J}^2, H] = 0,$$

where

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}.$$

The energy levels are thus $(2j+1)$-foldly degenerated.
Let's set the atom in magnetic field parallel to the z-axis.
The Hamiltonian is then appended by the term

$$Z = cS_z.$$

Now

$$[\boldsymbol{J}^2, S_z] \neq 0,$$

so the rotation symmetry is broken and the $(2j+1)$-fold degeneracy lifted.