Time reversal (reversal of motion)

The Newton equations of motion are invariant under the transformation $t \longrightarrow -t$: if $\boldsymbol{x}(t)$ is a solution of the equation

$$m\ddot{\boldsymbol{x}} = -\nabla V(\boldsymbol{x})$$

then also $\boldsymbol{x}(-t)$ is a solution.
At the moment $t = 0$ let there be a particle at the point $\boldsymbol{x}(t=0)$ with the momentum $\boldsymbol{p}(t=0)$. Then a particle at the same point but with the momentum $-\boldsymbol{p}(t=0)$ follows the trajectory $\boldsymbol{x}(-t)$.
In the quantum mechanical Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi,$$

due to the first derivative with respect to the time, $\psi(\boldsymbol{x},-t)$ is not a solution eventhough $\psi(\boldsymbol{x},t)$ were, but $\psi^*(\boldsymbol{x},-t)$ is. In quantum mechanics the time reversal has obviously something to do with the complex conjugation.
Let us consider the symmetry operation

$$|\alpha\rangle \longrightarrow |\tilde{\alpha}\rangle, \quad |\beta\rangle \longrightarrow |\tilde{\beta}\rangle.$$

We require that the absolute value of the scalar product is invariant under that operation:

$$|\langle\tilde{\beta}|\tilde{\alpha}\rangle| = |\langle\beta|\alpha\rangle|.$$

There are two possibilities to satisfy this condition:

1. $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle$, so the corresponding symmetry operator is unitary, that is

   $$\langle\beta|\alpha\rangle \longrightarrow \langle\beta|U^\dagger U|\alpha\rangle = \langle\beta|\alpha\rangle.$$

   The symmetries treated earlier have obeyed this condition.

2. $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^* = \langle\alpha|\beta\rangle$, so the symmetry operator cannot be unitary.

We define the *antiunitary* operator θ so that

$$\begin{aligned}\langle\tilde{\beta}|\tilde{\alpha}\rangle &= \langle\alpha|\beta\rangle^*\\ \theta(c_1|\alpha\rangle + c_2|\beta\rangle) &= c_1^*\theta|\alpha\rangle + c_2^*\theta|\beta\rangle,\end{aligned}$$

where

$$|\alpha\rangle \longrightarrow |\tilde{\alpha}\rangle = \theta|\alpha\rangle, \quad |\beta\rangle \longrightarrow |\tilde{\beta}\rangle = \theta|\beta\rangle$$

If the operator satisfies only the last condition it is called *antilinear*.
We define the complex conjugation operator K so that

$$Kc|\alpha\rangle = c^*K|\alpha\rangle.$$

We present the state $|\alpha\rangle$ in the base $\{|a'\rangle\}$. The effect of the operator K is then

$$\begin{aligned}|\alpha\rangle &= \sum_{a'}|a'\rangle\langle a'|\alpha\rangle \xrightarrow{K} |\tilde{\alpha}\rangle = \sum_{a'}\langle a'|\alpha\rangle^* K|a'\rangle\\ &= \sum_{a'}\langle a'|\alpha\rangle^*|a'\rangle.\end{aligned}$$

The fact that the operator K does not change the base states can be justified like:
The state $|a'\rangle$ represented in the base $\{|a'\rangle\}$ maps to the column vector

$$|a'\rangle \mapsto \begin{pmatrix}0\\0\\\vdots\\0\\1\\0\\\vdots\\0\end{pmatrix},$$

which is unaffected by the complex conjugation.
Note The effect of the operator K depends thus on the choice of the basis states.
If U is a unitary operator then the operator $\theta = UK$ is antiunitary.
Proof: Firstly

$$\begin{aligned}\theta(c_1|\alpha\rangle + c_2|\beta\rangle) &= UK(c_1|\alpha\rangle + c_2|\beta\rangle)\\ &= (c_1^*UK|\alpha\rangle + c_2^*UK|\beta\rangle)\\ &= (c_1^*\theta|\alpha\rangle + c_2^*\theta|\beta\rangle),\end{aligned}$$

so θ is antiliniear. Secondly, expanding the states $|\alpha\rangle$ and $|\beta\rangle$ in a complete basis $\{|a'\rangle\}$ we get

$$\begin{aligned}|\alpha\rangle \xrightarrow{\theta} |\tilde{\alpha}\rangle &= \sum_{a'}\langle a'|\alpha\rangle^* UK|a'\rangle\\ &= \sum_{a'}\langle a'|\alpha\rangle^* U|a'\rangle\\ &= \sum_{a'}\langle\alpha|a'\rangle U|a'\rangle\end{aligned}$$

and

$$|\tilde{\beta}\rangle = \sum_{a'}\langle a'|\beta\rangle^* U|a'\rangle \leftrightarrow \langle\tilde{\beta}| = \sum_{a'}\langle a'|\beta\rangle\langle a'|U^\dagger.$$

Thus the scalar product is

$$\begin{aligned}\langle\tilde{\beta}|\tilde{\alpha}\rangle &= \sum_{a''}\sum_{a'}\langle a''|\beta\rangle\langle a''|U^\dagger U|a'\rangle\langle\alpha|a'\rangle\\ &= \sum_{a'}\langle\alpha|a'\rangle\langle a'|\beta\rangle = \langle\alpha|\beta\rangle\\ &= \langle\beta|\alpha\rangle^*.\end{aligned}$$

The operator θ is thus indeed antiunitary. ■
Let Θ be the time reversal operator. We consider the transformation

$$|\alpha\rangle \longrightarrow \Theta|\alpha\rangle,$$

where $\Theta|\alpha\rangle$ is the time reversed (motion reversed) state.
If $|\alpha\rangle$ is the momentum eigenstate $|\boldsymbol{p}'\rangle$, we should have

$$\Theta|\boldsymbol{p}'\rangle = e^{i\varphi}|-\boldsymbol{p}'\rangle.$$

Let the system be at the moment $t = 0$ in the state $|\alpha\rangle$.
At a slightly later moment $t = \delta t$ it is in the state

$$|\alpha, t_0 = 0; t = \delta t\rangle = \left(1 - \frac{iH}{\hbar}\delta t\right)|\alpha\rangle.$$

We apply now, at the moment $t = 0$, the time reversal operator Θ and let the system evolve under the Hamiltonian H. Then at the moment δt the system is in the state

$$\left(1 - \frac{iH}{\hbar}\delta t\right)\Theta|\alpha\rangle.$$

If the motion of the system is invariant under time reversal this state should be the same as

$$\Theta|\alpha, t_0 = 0; -\delta t\rangle,$$

i.e. we first look at the state at the earlier moment $-\delta t$ and then reverse the direction of the momentum $\boldsymbol{p}$. Mathematically this condition can be expressed as

$$\left(1 - \frac{iH}{\hbar}\delta t\right)\Theta|\alpha\rangle = \Theta\left(1 - \frac{iH}{\hbar}(-\delta t)\right)|\alpha\rangle.$$

Thus we must have

$$-iH\Theta|\rangle = \Theta iH|\rangle,$$

where $|\rangle$ stands for an arbitrary state vector.
If Θ were linear we would obtain the anticommutator relation

$$H\Theta = -\Theta H.$$

If now $|n\rangle$ is an energy eigenstate corresponding to the eigenvalue E_n then, according to the anticommutation rule

$$H\Theta|n\rangle = -\Theta H|n\rangle = (-E_n)\Theta|n\rangle,$$

and the state $\Theta|n\rangle$ is an energy eigenstate corresponding to the eigenvalue $-E_n$. Thus most systems (those, whose energy spectrum is not bounded) would not have any ground state.
Thus the operator Θ must be antilinear, and, in order to be a symmetry operator, it must be antiunitary. Using the antilinearity for the right hand side of the condition

$$-iH\Theta|\rangle = \Theta iH|\rangle$$

we can write it as

$$\Theta iH|\rangle = -i\Theta H|\rangle.$$

So, we see that the operators commute:

$$\Theta H = H\Theta.$$

Note We have not defined the Hermitean conjugate of the antiunitary operator θ nor have we defined the meaning of the expression $\langle\beta|\theta$. That being, we let the time reversal operator Θ to operate always on the right and with the matrix element $\langle\beta|\Theta|\alpha\rangle$ we mean the expression $(\langle\beta|)\cdot(\Theta|\alpha\rangle)$.
Let $\otimes$ be an arbitrary linear operator. We define

$$|\gamma\rangle \equiv \otimes^\dagger|\beta\rangle,$$

so that

$$\langle\beta|\otimes = \langle\gamma|$$

and

$$\begin{aligned}\langle\beta|\otimes|\alpha\rangle &= \langle\gamma|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\gamma}\rangle \\ &= \langle\tilde{\alpha}|\Theta\otimes^\dagger|\beta\rangle = \langle\tilde{\alpha}|\Theta\otimes^\dagger\Theta^{-1}\Theta|\beta\rangle \\ &= \langle\tilde{\alpha}|\Theta\otimes^\dagger\Theta^{-1}|\tilde{\beta}\rangle.\end{aligned}$$

In partcular, for a Hermitean observable A we have

$$\langle\beta|A|\alpha\rangle = \langle\tilde{\alpha}|\Theta A\Theta^{-1}|\tilde{\beta}\rangle.$$

We say that the observable A is even or odd under time reversal depending on wheter in the equation

$$\Theta A\Theta^{-1} = \pm A$$

the upper or the lower sign holds. This together with the equation

$$\langle\beta|A|\alpha\rangle = \langle\tilde{\alpha}|\Theta A\Theta^{-1}|\tilde{\beta}\rangle$$

imposes certain conditions on the phases of the matrix elements of the operator A between the time reversed states. Namely, they has to satisfy

$$\langle\beta|A|\alpha\rangle = \pm\langle\tilde{\beta}|A|\tilde{\alpha}\rangle^*.$$

In particular, the expectation value satisfies the condition

$$\langle\alpha|A|\alpha\rangle = \pm\langle\tilde{\alpha}|A|\tilde{\alpha}\rangle.$$

Example The expectation value of the momentum operator $\boldsymbol{p}$.
We require that

$$\langle\alpha|\boldsymbol{p}|\alpha\rangle = -\langle\tilde{\alpha}|\boldsymbol{p}|\tilde{\alpha}\rangle,$$

so $\boldsymbol{p}$ is odd, or

$$\Theta\boldsymbol{p}\Theta^{-1} = -\boldsymbol{p}.$$

The momentum eigenstates satisfy

$$\begin{aligned}\boldsymbol{p}\Theta|\boldsymbol{p}'\rangle &= -\Theta\boldsymbol{p}\Theta^{-1}\Theta|\boldsymbol{p}'\rangle \\ &= (-\boldsymbol{p}')\Theta|\boldsymbol{p}'\rangle,\end{aligned}$$

i.e. $\Theta|\boldsymbol{p}'\rangle$ is the momentum eigenstates correponding to the eigenvalue $-\boldsymbol{p}'$:

$$\Theta|\boldsymbol{p}'\rangle = e^{i\varphi}|-\boldsymbol{p}'\rangle.$$

Similarly we can derive for the position operator $\boldsymbol{x}$ the expressions

$$\begin{aligned}\Theta\boldsymbol{x}\Theta^{-1} &= \boldsymbol{x} \\ \Theta|\boldsymbol{x}'\rangle &= |\boldsymbol{x}'\rangle\end{aligned}$$

when we impose the physically sensible condition

$$\langle\alpha|\boldsymbol{x}|\alpha\rangle = \langle\tilde{\alpha}|\boldsymbol{x}|\tilde{\alpha}\rangle.$$

We consider the basic commutation relations

$$[x_i, p_j]|\rangle = i\hbar\delta_{ij}|\rangle.$$

Now

$$\Theta[x_i, p_j]\Theta^{-1}\Theta|\rangle = \Theta i\hbar\delta_{ij}|\rangle,$$

from which, using the antilinearity and the time reversal properties of the operators $\boldsymbol{x}$ and $\boldsymbol{p}$ we get

$$[x_i, (-p_j)]\Theta|\rangle = -i\hbar\delta_{ij}\Theta|\rangle.$$

We see thus that the commutation rule

$$[x_i, p_j]|\rangle = i\hbar\delta_{ij}|\rangle$$

remains invariant under the time reversal.
Correspondingly, the requirement of the invariance of the commutation rule

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$$

leads to the condition

$$\Theta\boldsymbol{J}\Theta^{-1} = -\boldsymbol{J}.$$

This agrees with transformation properties of the orbital angular momentum $\boldsymbol{x}\times\boldsymbol{p}$.

Wave functions
We expand the state $|\alpha\rangle$ with the help of position eigenstates:

$$|\alpha\rangle = \int d^3x'\,|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|\alpha\rangle.$$

Now

$$\begin{aligned}\Theta|\alpha\rangle &= \int d^3x'\,\Theta|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|\alpha\rangle^* \\ &= \int d^3x'\,|\boldsymbol{x}'\rangle\langle\boldsymbol{x}'|\alpha\rangle^*,\end{aligned}$$

so under the time reversal the wave function

$$\psi(\boldsymbol{x}') = \langle\boldsymbol{x}'|\alpha\rangle$$

transforms like

$$\psi(\boldsymbol{x}') \longrightarrow \psi^*(\boldsymbol{x}').$$

If in particular we have

$$\psi(\boldsymbol{x}') = R(r)Y_l^m(\theta,\phi),$$

we see that

$$Y_l^m(\theta,\phi) \longrightarrow Y_l^{m*}(\theta,\phi) = (-1)^m Y_l^{-m}(\theta,\phi).$$

Because Y_l^m is the wave function belonging to the state $|lm\rangle$ we must have

$$\Theta|lm\rangle = (-1)^m|l,-m\rangle.$$

The probability current corresponding to the wave function $R(r)Y_l^m$ seems to turn clockwise when looked at from the direction of the positive z-axis and $m>0$. The probability current of the corresponding time reversed state on the other hand turns counterclockwise because m changes its sign under the operation.
The spinles particles obey
Theorem 1 *If the Hamiltonian H is invariant under the time reversal and the energy eigenstate $|n\rangle$ nondegenerate then the corresponding energy eigenfunction is real (or more generally a real function times a phase factor independent on the coordinate $\boldsymbol{x}'$).*

$$H\Theta|n\rangle = \Theta H|n\rangle = E_n\Theta|n\rangle,$$

so the states $|n\rangle$ and $\Theta|n\rangle$ have the same energy. Because the state $|n\rangle$ was supposed to be nondegenerate they must represent the same state. The wave function of the state $|n\rangle$ is $\langle\boldsymbol{x}'|n\rangle$ and the one of the state $\Theta|n\rangle$ correspondingly $\langle\boldsymbol{x}'|n\rangle^*$. These must be same (or more accurately, they can differ only by a phase factor which does not depend on the coordinate $\boldsymbol{x}'$), i.e.

$$\langle\boldsymbol{x}'|n\rangle = \langle\boldsymbol{x}'|n\rangle^* \quad \blacksquare$$

For example the wave function of a nondegenerate groundstate is always real.
For a spinles particle in the state $|\alpha\rangle$ we get

$$\begin{aligned}\Theta|\alpha\rangle &= \Theta\int d\boldsymbol{x}'\,\langle\boldsymbol{x}'|\alpha\rangle|\boldsymbol{x}'\rangle \\ &= \int d\boldsymbol{x}'\,\langle\boldsymbol{x}'|\alpha\rangle^*|\boldsymbol{x}'\rangle = K|\alpha\rangle,\end{aligned}$$

i.e. the time reversal is equivalent to the complex conjugation.
On the other hand, in the momentum space we have

$$\begin{aligned}\Theta|\alpha\rangle &= \int d^3p'\,|-\boldsymbol{p}'\rangle\langle\boldsymbol{p}'|\alpha\rangle^* \\ &= \int d^3p'\,|\boldsymbol{p}'\rangle\langle-\boldsymbol{p}'|\alpha\rangle^*,\end{aligned}$$

because

$$\Theta|\boldsymbol{p}'\rangle = |-\boldsymbol{p}'\rangle.$$

The momentum space wave function transform thus under time reversal like

$$\phi(\boldsymbol{p}') \longrightarrow \phi^*(-\boldsymbol{p}').$$

We consider a spin $\frac{1}{2}$ particle the spin of which is oriented along $\hat{\boldsymbol{n}}$. The corresponding state is obtained by rotating the state $|S_z;\uparrow\rangle$:

$$|\boldsymbol{n};\uparrow\rangle = e^{-iS_z\alpha/\hbar}e^{-iS_y\beta/\hbar}|S_z;\uparrow\rangle,$$

where α and β are the direction angles of the vector $\hat{\boldsymbol{n}}$.
Because

$$\Theta\boldsymbol{J}\Theta^{-1} = -\boldsymbol{J}.$$

we see that

$$\Theta|\boldsymbol{n};\uparrow\rangle = e^{-iS_z\alpha/\hbar}e^{-iS_y\beta/\hbar}\Theta|S_z;\uparrow\rangle.$$

Furthermore, due to the oddity of the angular momentum, it follows that

$$J_z\Theta|S_z;\uparrow\rangle = -\frac{\hbar}{2}\Theta|S_z;\uparrow\rangle,$$

so we must have

$$\Theta|S_z;\uparrow\rangle = \eta|S_z;\downarrow\rangle,$$

where η is an arbitrary phase factor. So we get

$$\Theta|\boldsymbol{n};\uparrow\rangle = \eta|\boldsymbol{n};\downarrow\rangle.$$

On the other hand we have

$$|\boldsymbol{n};\downarrow\rangle = e^{-i\alpha S_z/\hbar}e^{-i(\pi+\beta)S_y/\hbar}|S_z;\uparrow\rangle,$$

so

$$\begin{aligned}\eta|\boldsymbol{n};\downarrow\rangle &= \Theta|\boldsymbol{n};\uparrow\rangle = e^{-iS_z\alpha/\hbar}e^{-iS_y\beta/\hbar}\Theta|S_z;\uparrow\rangle \\ &= \eta e^{-i\alpha S_z/\hbar}e^{-i(\pi+\beta)S_y/\hbar}|S_z;\uparrow\rangle.\end{aligned}$$

Writing

$$\Theta = UK,\ U \text{ unitary}$$

and recalling that the complex conjugation K has no effect on the base states we see that

$$\Theta = \eta e^{-i\pi S_y/\hbar}K = -i\eta\left(\frac{2S_y}{\hbar}\right)K.$$

Now

$$\begin{aligned}e^{-i\pi S_y/\hbar}|S_z;\uparrow\rangle &= +|S_z;\downarrow\rangle \\ e^{-i\pi S_y/\hbar}|S_z;\downarrow\rangle &= -|S_z;\uparrow\rangle,\end{aligned}$$

so the effect of the time reversal on a general spin $\frac{1}{2}$ state is

$$\Theta(c_\uparrow|S_z;\uparrow\rangle + c_\downarrow|S_z;\downarrow\rangle) = +\eta c_\uparrow^*|S_z;\downarrow\rangle - \eta c_\downarrow^*|S_z;\uparrow\rangle.$$

Applying the operator Θ once again we get

$$\begin{aligned}&\Theta^2(c_\uparrow|S_z;\uparrow\rangle + c_\downarrow|S_z;\downarrow\rangle) \\ &\quad = -|\eta|^2c_\uparrow|S_z;\uparrow\rangle - |\eta|^2c_\downarrow|S_z;\downarrow\rangle \\ &\quad = -(c_\uparrow|S_z;\uparrow\rangle + c_\downarrow|S_z;\downarrow\rangle),\end{aligned}$$

i.e. for an arbitrary spin orientation we have

$$\Theta^2 = -1.$$

From the relation

$$\Theta|lm\rangle = (-1)^m|l,-m\rangle$$

we see that for spinles particles we have

$$\Theta^2 = 1.$$

In general, one can show that

$$\begin{aligned}\Theta^2|j\text{ half integer}\rangle &= -|j\text{ half integer}\rangle \\ \Theta^2|j\text{ integer}\rangle &= +|j\text{ integer}\rangle.\end{aligned}$$

Generally we can write

$$\Theta = \eta e^{-i\pi J_y/\hbar}K.$$

Now

$$e^{-2i\pi J_y/\hbar}|jm\rangle = (-1)^{2j}|jm\rangle,$$

so

$$\begin{aligned}\Theta^2|jm\rangle &= \Theta\left(\eta e^{-i\pi J_y/\hbar}|jm\rangle\right) \\ &= |\eta|^2e^{-2i\pi J_y/\hbar}|jm\rangle \\ &= (-1)^{2j}|jm\rangle.\end{aligned}$$

Thus we must have

$$\Theta^2 = (-1)^{2j}.$$

Often one chooses

$$\Theta|jm\rangle = i^{2m}|j,-m\rangle.$$

Spherical tensors

Let us suppose that the operator A is either even or odd, i.e.

$$\Theta A\Theta^{(-1)} = \pm A.$$

We saw that then we have

$$\langle\alpha|A|\alpha\rangle = \pm\langle\tilde{\alpha}|A|\tilde{\alpha}\rangle.$$

In an eigenstate of the angular momentum we have thus

$$\langle\alpha,jm|A|\alpha,jm\rangle = \pm\langle\alpha,j,-m|A|\alpha,j,-m\rangle.$$

Let now A be a component of a Hermitian spherical tensor:

$$A = T_q^{(k)}.$$

According to the Wigner-Eckart theorem it is sufficient to consider only the component $q=0$.

We define $T^{(k)}$ to be even/odd under the time reversal if

$$\Theta T_{q=0}^{(k)}\Theta^{-1} = \pm T_{q=0}^{(k)}.$$

Then we have

$$\langle\alpha,jm|T_0^{(k)}|\alpha,jm\rangle = \pm\langle\alpha,j,-m|T_0^{(k)}|\alpha,j,-m\rangle.$$

The state $|\alpha,j,-m\rangle$ is obtained by rotating the state $|\alpha,jm\rangle$:

$$\mathcal{D}(0,\pi,0)|\alpha,jm\rangle = e^{i\varphi}|\alpha,j,-m\rangle.$$

On the other hand, due to the definition of the spherical tensor

$$\mathcal{D}^\dagger(R)T_q^{(k)}\mathcal{D}(R) = \sum_{q'=-k}^{k}\mathcal{D}_{qq'}^{(k)^*}(R)T_{q'}^{(k)},$$

we get

$$\mathcal{D}^\dagger(0,\pi,0)T_0^{(k)}\mathcal{D}(0,\pi,0) = \sum_q\mathcal{D}_{0q}^{(k)}(0,\pi,0)T_q^{(k)}.$$

Now

$$\mathcal{D}_{00}^{(k)}(0,\pi,0) = P_k(\cos\pi) = (-1)^k,$$

so we have

$$\begin{aligned}&\mathcal{D}^\dagger(0,\pi,0)T_0^{(k)}\mathcal{D}(0,\pi,0) \\ &\quad = (-1)^kT_0^{(k)} + (q\neq 0\text{ components}).\end{aligned}$$

Furthermore

$$\langle\alpha,jm|T_{q\neq 0}^{(k)}|\alpha,jm\rangle = 0,$$

since the m selection rule would require $m = m + q$. So we get

$$
\begin{aligned}
&\langle \alpha, jm | T_0^{(k)} | \alpha, jm \rangle \\
&\quad = \pm \langle \alpha, jm | \mathcal{D}^\dagger(0, \pi, 0) T_0^{(k)} \mathcal{D}(0, \pi, 0) | \alpha, jm \rangle \\
&\quad = \pm (-1)^k \langle \alpha, jm | T_0^{(k)} | \alpha, jm \rangle .
\end{aligned}
$$

Note Unlike under other symmetries the invariance of the Hamiltonian under the time reversal

$$[\Theta, H] = 0,$$

does not lead to any conservation laws. This is due to the fact that the time evolution operator is not invariant:

$$\Theta U(t, t_0) \neq U(t, t_0) \Theta.$$

Time reversal and degeneracy
Let us suppose that

$$[\Theta, H] = 0.$$

Then the energy eigenstates obey

$$
\begin{aligned}
H|n\rangle &= E_n|n\rangle \\
H\Theta|n\rangle &= E_n\Theta|n\rangle .
\end{aligned}
$$

If we now had

$$\Theta|n\rangle = e^{i\delta}|n\rangle,$$

then, reapplying the time reversal we would obtain

$$\Theta^2|n\rangle = e^{-i\delta}\Theta|n\rangle = |n\rangle,$$

or

$$\Theta^2 = 1.$$

This is, however, impossible if the system j is half integer, because then $\Theta^2 = -1$. In systems of this kind $|n\rangle$ and $\Theta|n\rangle$ are degenerate.
Example Electon in electromagnetic field
If a particle is influenced by an external static electric field

$$V(\boldsymbol{x}) = e\phi(\boldsymbol{x}),$$

then clearly the Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x})$$

is invariant under the time reversal:

$$[\Theta, H] = 0.$$

If now there are odd number of electrons in the system the total j is half integer. Thus, in a system of this kind there is at least twofold degeneracy, so called *Kramers' degeneracy.*
In the magnetic field

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

the Hamiltonian of an electron contains such terms as

$$\boldsymbol{S} \cdot \boldsymbol{B}, \quad \boldsymbol{p} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \boldsymbol{p}.$$

The magnetic field $\boldsymbol{B}$ is external, independent on the system, so

$$[\Theta, \boldsymbol{B}] = 0 \text{ ja } [\Theta, \boldsymbol{A}] = 0.$$

On the other hand, $\boldsymbol{S}$ and $\boldsymbol{p}$ are odd, or

$$\Theta\boldsymbol{S}\Theta^{-1} = -\boldsymbol{S} \text{ ja } \Theta\boldsymbol{p}\Theta^{-1} = -\boldsymbol{p},$$

so

$$[\Theta, H] \neq 0.$$

We say that magnetic field breaks the time reversal symmetry and lifts the Kramers degeneracy.