Time dependent potentials

We have solved the problem

$$H_0|n\rangle = E_n|n\rangle$$

completely and want to solve the eigenstates of the Hamiltonian

$$H = H_0 + V(t).$$

Since the Hamiltonian depends on time we have

$$\mathcal{U} \neq e^{-iHt/\hbar},$$

so a system in a stationary state $|i\rangle$ can in the course of time get components also in other stationary states.

Pictures of the time evolution
At the moment $t = 0$ let the system be in the state

$$|\alpha\rangle = \sum_n c_n(0)|n\rangle$$

and at the moment t in the state

$$|\alpha, t_0 = 0; t\rangle = \sum_n c_n(t)e^{-iE_nt/\hbar}|n\rangle.$$

Note The time dependence of the coefficients $c_n(t)$ is due only to the potential $V(t)$. The effect of the Hamiltonian H_0 is in the phase factors $e^{-iE_nt/\hbar}$.

Schrödinger's picture
The evolution of the state vectors is governed by the time evolution operator:

$$|\alpha, t_0 = 0; t\rangle_S = \mathcal{U}(t)|\alpha, t_0 = 0\rangle.$$

Heisenberg's picture
The state vectors remain constant, i.e.

$$|\alpha, t_0 = 0, t\rangle_H = |\alpha, t_0 = 0\rangle.$$

On the other hand, the operators depend on time. We can go from the time independent operators of the Schrödinger picture to the operators of the Heisenberg picture via the transformation

$$A_H(t) = \mathcal{U}^\dagger(t)A_S\mathcal{U}(t).$$

If the Hamiltonian does not depend on time then

$$H_H(t) = \mathcal{U}^\dagger(t)H\mathcal{U}(t) = H$$

and

$$\frac{dA_H}{dt} = \frac{1}{i\hbar}[A_H, H_H] = \frac{1}{i\hbar}[A_H, H].$$

Interaction picture
The state vectors depend on time as

$$|\alpha, t_0; t\rangle_I \equiv e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S.$$

At the moment $t = 0$ we have obviously

$$|\rangle_S = |\rangle_I = |\rangle_H.$$

The interaction picture observables A_I are defined so that

$$A_I \equiv e^{iH_0t/\hbar}A_Se^{-iH_0t/\hbar}.$$

In particular we have

$$V_I = e^{iH_0t/\hbar}Ve^{-iH_0t/\hbar}.$$

We see that the equation governing the time dpendence of the interaction picture state vectors is

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}&|\alpha, t_0; t\rangle_I \\
&= i\hbar\frac{\partial}{\partial t}\left(e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S\right) \\
&= -H_0e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S \\
&\quad + e^{iH_0t/\hbar}(H_0 + V)|\alpha, t_0; t\rangle_S \\
&= e^{iH_0t/\hbar}Ve^{-iH_0t/\hbar}e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S,
\end{aligned}$$

so

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = V_I|\alpha, t_0; t\rangle_I.$$

If now A_S does not depend on time we can derive

$$\frac{dA_I}{dt} = \frac{1}{i\hbar}[A_I, H_0],$$

which in turn resembles the Heisenberg equation of motion provided that in the latter we substitute for the total Hamiltonian H the stationary operator H_0.
We expand state vectors in the base $\{|n\rangle\}$:

$$|\alpha, t_0; t\rangle_I = \sum_n c_n(t)|n\rangle.$$

If now $t_0 = 0$ so multiplying the previous expansion

$$|\alpha, t_0 = 0; t\rangle = \sum_n c_n(t)e^{-iE_nt/\hbar}|n\rangle$$

on both sides by the operator $e^{-iH_0t/\hbar}$ we get

$$|\alpha, t_0 = 0; t\rangle_I = \sum_n c_n(t)|n\rangle,$$

i.e. the coefficients c_n are equal. We just derived the equation

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = V_I|\alpha, t_0; t\rangle_I.$$

From this we get

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\langle n|\alpha, t_0; t\rangle_I &= \langle n|V_I|\alpha, t_0; t\rangle_I \\
&= \sum_m \langle n|V_I|m\rangle\langle m|\alpha, t_0; t\rangle_I.
\end{aligned}$$

The matrix elements of the operator V_I are

$$\begin{aligned}
\langle n|V_I|m\rangle &= \langle n|e^{iH_0t/\hbar}Ve^{-iH_0t/\hbar}|m\rangle \\
&= V_{nm}(t)e^{i(E_n-E_m)t/\hbar}
\end{aligned}$$

.

Because we furthermore have

$$c_n(t) = \langle n|\alpha, t_0; t\rangle_I,$$

we can write the equation governing the time dependence of the superposition coefficients as

$$i\hbar\frac{d}{dt}c_n(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t),$$

where

$$\omega_{nm} \equiv \frac{E_n - E_m}{\hbar} = -\omega_{mn}.$$

This system of differential equations can be written explicitely in the matrix form

$$i\hbar\begin{pmatrix}\dot{c}_1\\ \dot{c}_2\\ \dot{c}_3\\ \vdots\end{pmatrix} = \begin{pmatrix} V_{11} & V_{12}e^{i\omega_{12}t} & & \cdots\\ V_{21}e^{i\omega_{21}t} & V_{22} & & \cdots\\ & & V_{33} & \cdots\\ \vdots & \vdots & & \ddots\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\\ \vdots\end{pmatrix}.$$

Example Two state systems.
Suppose that

$$H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| \quad (E_1 < E_2)$$

and that the time dependent potential is like

$$V(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|.$$

The matrix elements V_{nm} are now

$$\begin{aligned} V_{12} &= V_{21}^* = \gamma e^{i\omega t}\\ V_{11} &= V_{22} = 0,\end{aligned}$$

so transitions between the states $|1\rangle$ and $|2\rangle$ are possible. The system of differential equations to be solved is

$$\begin{aligned} i\hbar\dot{c}_1 &= \gamma e^{i\omega t}e^{i\omega_{12}t}c_2\\ i\hbar\dot{c}_2 &= \gamma e^{-i\omega t}e^{i\omega_{21}t}c_1,\end{aligned}$$

where

$$\omega_{21} = -\omega_{12} = \frac{E_2 - E_1}{\hbar}.$$

We can see that the solution satisfying the initial conditions

$$c_1(0) = 1, \quad c_2(0) = 0$$

is

$$\begin{aligned} |c_2(t)|^2 &= \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4}\\ &\qquad \times \sin^2\left\{\left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega-\omega_{21})^2}{4}\right]^{1/2} t\right\}\\ |c_1(t)|^2 &= 1 - |c_2(t)|^2.\end{aligned}$$

The system oscillates between the states $|1\rangle$ and $|2\rangle$ with the angular velocity

$$\Omega = \sqrt{\left(\frac{\gamma^2}{\hbar^2}\right) + \frac{(\omega-\omega_{21})^2}{4}}.$$

The amplitude of the oscillations is at its maximum, i.e. we are in a resonance, when

$$\omega \approx \omega_{21} = \frac{E_2 - E_1}{\hbar}.$$

Example Spin magnetic resonance.
We put a spin $\frac{1}{2}$ particle into

- time independent magnetic field parallel to the z axis,
- time dependent magnetic field rotating in the xy plane,

i.e.

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + B_1(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t)$$

when the fields B_0 and B_1 are constant. Since the magnetic moment of the electron is

$$\boldsymbol{\mu} = \frac{e}{m_e c}\boldsymbol{S},$$

the Hamiltonian is the sum of the terms

$$\begin{aligned} H_0 &= -\left(\frac{e\hbar B_0}{2m_e c}\right)(|S_z;\uparrow\rangle\langle S_z;\uparrow| - |S_z;\downarrow\rangle\langle S_z;\downarrow|)\\ V(t) &= -\left(\frac{e\hbar B_1}{2m_e c}\right)\\ &\quad \times\big[\cos\omega t(|S_z;\uparrow\rangle\langle S_z;\downarrow| + |S_z;\downarrow\rangle\langle S_z;\uparrow|)\\ &\quad + \sin\omega t(-i|S_z;\uparrow\rangle\langle S_z;\downarrow| + i|S_z;\downarrow\rangle\langle S_z;\uparrow|)\big].\end{aligned}$$

If $e < 0$, then

$$E_2 = E_\uparrow = \frac{|e|\hbar B_0}{2m_e c} > E_1 = E_\downarrow = -\frac{|e|\hbar B_0}{2m_e c}.$$

We can thus identify in the above treated two stated system the quantities:

$$\begin{aligned} |S_z;\uparrow\rangle &\mapsto |2\rangle \quad \text{(higher state)}\\ |S_z;\downarrow\rangle &\mapsto |1\rangle \quad \text{(lower state)}\\ \frac{|e|B_0}{m_e c} &\mapsto \omega_{21}\\ -\frac{e\hbar B_1}{2m_e c} &\mapsto \gamma, \quad \omega\mapsto\omega.\end{aligned}$$

Comparing with our earlier discussion on the spin precession we see that

- if $B_1 = 0$ and $B_0 \neq 0$, the the expectation value $\langle S_{x,y}\rangle$ rotates in the course of time counterclockwise but the probabilities $|c_1|^2$ and $|c_2|^2$ remain still constant.
- if $B_1 \neq 0$, the the coefficients $|c_1|^2$ and $|c_2|^2$ are functions of time.

When the resonance condition

$$\omega \approx \omega_{21}$$

holds the probability for the spin flips

$$|S_z;\uparrow\rangle \longleftrightarrow |S_z;\downarrow\rangle$$

is very high.
Because experimental production of rotating magnetic fields is difficult it is common to use a field oscillating for example along the x axis. This can be divided into components rotating counterclockwise and clockwise:

$$\begin{aligned} &2B_1\hat{\boldsymbol{x}}\cos\omega t \\ &\quad = B_1(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t) \\ &\quad\quad + B_1(\hat{\boldsymbol{x}}\cos\omega t - \hat{\boldsymbol{y}}\sin\omega t). \end{aligned}$$

In experiments one usually has

$$\frac{B_1}{B_0} \ll 1,$$

so

$$\frac{\gamma}{\hbar} = \frac{|e|B_1}{4m_e c} = \frac{|e|B_0}{m_e c}\frac{B_1}{4B_0} = \omega_{21}\frac{B_1}{4B_0} \ll \omega_{21}.$$

If now the component rotating counterclockwise triggers the resonance condition

$$\omega \approx \omega_{21},$$

the the transition probability due to this component is

$$\begin{aligned} |c_R(t)|^2 &= \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega-\omega_{21})^2/4} \\ &\quad \times \sin^2\left\{\left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega-\omega_{21})^2}{4}\right]^{1/2} t\right\} \\ &\approx \sin^2\left(\frac{\gamma}{\hbar}t\right). \end{aligned}$$

The clockwise rotating component,

$$\omega = -\omega_{21},$$

contributes

$$\begin{aligned} |c_L(t)|^2 &\approx \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + \omega_{21}^2} \\ &\quad \times \sin^2\left\{\left[\frac{\gamma^2}{\hbar^2} + \omega_{21}^2\right]^{1/2} t\right\} \\ &\ll |c_R(t)|^2. \end{aligned}$$