Relativistic quantum mechanics

Classical fields

We suppose that the Lagrange function

$$L = L(q_i, \dot{q}_i) = T - V$$

of classical mechanics does not depend explicitely on time. From the Hamilton variation principle

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt = 0, \quad \delta q_i(t)\big|_{t=t_{1,2}} = 0$$

one can derive the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.$$

The Hamiltonian function of the Hamiltonian mechanics is

$$H = \sum_i p_i \dot{q}_i - L,$$

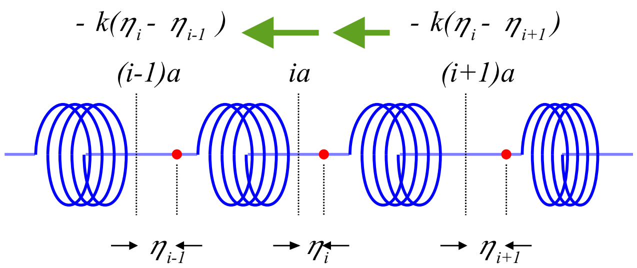
where the canonically conjugated momentum p_i of q_i is

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

The equations of motion are now

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}. \end{aligned}$$

Many body system



We consider N identical particles coupled to eachother by identical parallel springs. The Lagrangian of the system is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}\sum_i^N \left[m\dot{\eta}_i^2 - k(\eta_{i+1} - \eta_i)^2\right], \end{aligned}$$

when η_i is the deviation of the particle i from its equilibrium position ia.
We write this as

$$\begin{aligned} L &= \frac{1}{2}\sum_i^N \left[m\dot{\eta}_i^2 - k(\eta_{i+1} - \eta_i)^2\right] \\ &= \sum_i^N a\frac{1}{2}\left[\frac{m}{a}\dot{\eta}_i^2 - ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2\right] \\ &= \sum_i^N a\mathcal{L}_i. \end{aligned}$$

Here $\mathcal{L}_i$ is the linear Lagrangian density.
We go to continuum by substituting the limits

$$\begin{aligned} a &\longrightarrow dx \\ \frac{m}{a} &\longrightarrow \mu = \text{linear mass density} \\ \frac{\eta_{i+1} - \eta_i}{a} &\longrightarrow \frac{\partial \eta}{\partial x} \\ ka &\longrightarrow Y = \text{Young's modulus.} \end{aligned}$$

Now

$$L = \int \mathcal{L}\, dx,$$

where

$$\mathcal{L} = \frac{1}{2}\left[\mu\dot{\eta}^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2\right].$$

In the continuous case the Hamiltonian variation principle takes the form

$$\begin{aligned} &\delta \int_{t_1}^{t_2} L\, dt \\ &= \delta \int_{t_1}^{t_2} dt \int dx\, \mathcal{L}\left(\eta, \dot{\eta}, \frac{\partial \eta}{\partial x}\right) \\ &= \int dt \int dx \left\{\frac{\partial \mathcal{L}}{\partial \eta}\delta\eta + \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)}\delta\left(\frac{\partial \eta}{\partial x}\right)\right. \\ &\quad \left. + \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)}\delta\left(\frac{\partial \eta}{\partial t}\right)\right\} \\ &= \int dt \int dx \left\{\frac{\partial \mathcal{L}}{\partial \eta} + \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)}\frac{\partial}{\partial x}(\delta\eta)\right. \\ &\quad \left. + \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)}\frac{\partial}{\partial t}(\delta\eta)\right\} \\ &= \int dt \int dx \left\{\frac{\partial \mathcal{L}}{\partial \eta} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)}\right)\right. \\ &\quad \left. - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)}\right)\right\}\delta\eta. \end{aligned}$$

To get the variation to vanish for all $\delta\eta$ one must satisfy the Euler-Lagrange equation

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)} + \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)} - \frac{\partial \mathcal{L}}{\partial \eta} = 0.$$

When

$$\mathcal{L} = \frac{1}{2}\left[\mu\dot{\eta}^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2\right],$$

then

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \eta} &= 0 \\ \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)} &= -Y\frac{\partial}{\partial x}\frac{\partial \eta}{\partial x} = -Y\frac{\partial^2 \eta}{\partial x^2} \\ \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)} &= \mu\frac{\partial^2 \eta}{\partial t^2}. \end{aligned}$$

Substituting into the Euler-Lagrange equation we get

$$Y\frac{\partial^2 \eta}{\partial x^2} - \mu\frac{\partial^2 \eta}{\partial t^2} = 0,$$

which describes a wave progating in one dimension with the velocity $\sqrt{Y/\mu}$.
We define the canonically conjugated momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\eta}}$$

and the Hamiltonian density

$$\mathcal{H} = \dot{\eta}\pi - \mathcal{L}.$$

Now

$$\pi = \mu\dot{\eta},$$

so

$$\begin{aligned}\mathcal{H} &= \mu\dot{\eta}^2 - \frac{1}{2}\left[\mu\dot{\eta}^2 - Y\left(\frac{\partial\eta}{\partial x}\right)^2\right] \\ &= \frac{1}{2}\mu\dot{\eta}^2 + \frac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2.\end{aligned}$$

The Lagrangian formalism generalizes easily to three dimensions. The Euler-Lagrange equation takes then the form

$$\sum_{k=1}^{3}\frac{\partial}{\partial x_k}\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_k)} + \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial t)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

Covariant formulation
We employ the metrics where the components of a four-vector b_μ are

$$b_\mu = (b_1, b_2, b_3, b_4) = (\boldsymbol{b}, ib_0).$$

In particular the coordinate four-vector is

$$x_\mu = (x_1, x_2, x_3, x_4) = (\boldsymbol{x}, ict).$$

Under Lorentz transformations the coordinate vector transforms according to the equation

$$x'_\mu = a_{\mu\nu}x_\nu.$$

The coefficients of the Lorentz transformation satisfy the orthogonality condition

$$a_{\mu\nu}a_{\mu\lambda} = \delta_{\nu\lambda}, \quad (a^{-1})_{\mu\nu} = a_{\nu\mu},$$

so that

$$x_\mu = (a^{-1})_{\mu\nu}x'_\nu = a_{\nu\mu}x'_\nu.$$

We define the *four-vector* so that under Lorentz transformations it transforms like x_μ.
Now

$$\frac{\partial}{\partial x'_\mu} = \frac{\partial x_\nu}{\partial x'_\mu}\frac{\partial}{\partial x_\nu} = a_{\mu\nu}\frac{\partial}{\partial x_\nu},$$

so the four-gradient $\partial/\partial x_\mu$ is a four-vector.
The scalar product of the four-vectors b and c,

$$\begin{aligned}b\cdot c &= b_\mu c_\mu = \sum_{j=1}^{3} b_j c_j + b_4 c_4 \\ &= \boldsymbol{b}\cdot\boldsymbol{c} - b_0 c_0,\end{aligned}$$

is invariant under Lorentz transformations.
Using the four-vector notation the Euler-Lagrange equation can be written into the compact form

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_\mu)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

We see that the field equation derived from the Lagrangian density $\mathcal{L}$ is covariant provided that the Lagrangian density $\mathcal{L}$ itself is relativistically scalar (invariant under Lorentz transformations).