Dirac's equation

We construct relativistically covariant equation that takes into account also the spin.

The kinetic energy operator is

$$H^{(\mathrm{KE})} = \frac{\boldsymbol{p}^2}{2m}.$$

Previously we derived for Pauli spin matrices the relation

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})^2 = |\boldsymbol{a}|^2,$$

so we can also write

$$H^{(\mathrm{KE})} = \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{2m}.$$

However, when the particle moves under the influence of a vector potential these expressions differ. Substituting

$$\boldsymbol{p} \mapsto \boldsymbol{p} - e\boldsymbol{A}/c$$

the latter takes the form

$$\begin{aligned}
&\frac{1}{2m}\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{e\boldsymbol{A}}{c}\right)\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{e\boldsymbol{A}}{c}\right)\\
&\quad= \frac{1}{2m}\left(\boldsymbol{p}-\frac{e\boldsymbol{A}}{c}\right)^2\\
&\quad\quad+\frac{i}{2m}\boldsymbol{\sigma}\cdot\left[\left(\boldsymbol{p}-\frac{e\boldsymbol{A}}{c}\right)\times\left(\boldsymbol{p}-\frac{e\boldsymbol{A}}{c}\right)\right]\\
&\quad= \frac{1}{2m}\left(\boldsymbol{p}-\frac{e\boldsymbol{A}}{c}\right)^2 - \frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B},
\end{aligned}$$

where we have used the identities

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b})$$

and

$$\boldsymbol{p}\times\boldsymbol{A} = -i\hbar(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\times\boldsymbol{p}.$$

Let us suppose that for the relativistically invariant expression

$$(E^2/c^2) - \boldsymbol{p}^2 = (mc)^2$$

the operator analogy

$$\frac{1}{c^2}{E^{(\mathrm{op})}}^2 - \boldsymbol{p}^2 = (mc)^2$$

holds. Here

$$E^{(\mathrm{op})} = i\hbar\frac{\partial}{\partial t} = i\hbar c\frac{\partial}{\partial x_0}$$

and

$$\boldsymbol{p} = -i\hbar\nabla.$$

We write the operator equation into the form

$$\left(\frac{E^{(\mathrm{op})}}{c} - \boldsymbol{\sigma}\cdot\boldsymbol{p}\right)\left(\frac{E^{(\mathrm{op})}}{c} + \boldsymbol{\sigma}\cdot\boldsymbol{p}\right) = (mc)^2$$

or

$$\left(i\hbar\frac{\partial}{\partial x_0} + \boldsymbol{\sigma}\cdot i\hbar\nabla\right)\left(i\hbar\frac{\partial}{\partial x_0} - \boldsymbol{\sigma}\cdot i\hbar\nabla\right)\phi = (mc)^2\phi.$$

Here ϕ is a two component wave function (spinor).

We define new two component wave functions

$$\begin{aligned}
\phi^{(R)} &= \frac{1}{mc}\left(i\hbar\frac{\partial}{\partial x_0} - i\hbar\boldsymbol{\sigma}\cdot\nabla\right)\phi\\
\phi^{(L)} &= \phi.
\end{aligned}$$

It is easy to see that these satisfy the set of simultaneous equations

$$\begin{aligned}
\left[i\hbar\boldsymbol{\sigma}\cdot\nabla - i\hbar\frac{\partial}{\partial x_0}\right]\phi^{(L)} &= -mc\phi^{(R)}\\
\left[-i\hbar\boldsymbol{\sigma}\cdot\nabla - i\hbar\frac{\partial}{\partial x_0}\right]\phi^{(R)} &= -mc\phi^{(L)}.
\end{aligned}$$

We define yet new two component wave functions

$$\begin{aligned}
\psi_A &= \phi^{(R)} + \phi^{(L)}\\
\psi_B &= \phi^{(R)} - \phi^{(L)}.
\end{aligned}$$

These in turn satisfy the matrix equation

$$\begin{pmatrix} -i\hbar\frac{\partial}{\partial x_0} & -i\hbar\boldsymbol{\sigma}\cdot\nabla \\ i\hbar\boldsymbol{\sigma}\cdot\nabla & i\hbar\frac{\partial}{\partial x_0}\end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = -mc\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}.$$

We now define the four component wave function

$$\psi = \begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = \begin{pmatrix}\phi^{(R)}+\phi^{(L)}\\ \phi^{(R)}-\phi^{(L)}\end{pmatrix}$$

and the 4×4-matrices

$$\begin{aligned}
\gamma_k &= \begin{pmatrix} 0 & -i\sigma_k\\ i\sigma_k & 0\end{pmatrix}\\
\gamma_4 &= \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.
\end{aligned}$$

We end up with the *Dirac's equation*

$$\left(\boldsymbol{\gamma}\cdot\nabla + \gamma_4\frac{\partial}{\partial(ix_0)}\right)\psi + \frac{mc}{\hbar}\psi = 0$$

for free spin-$\frac{1}{2}$ particles. Employing the four vector notation the equation takes the form

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi = 0.$$

Note The Dirac equation is in fact a set of *four* coupled linear differential equations. The wave function ψ is the four component vector

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}.$$

This kind of a four component object is called *bispinor* or Dirac's spinor. Explicitely written down the Dirac equation is

$$\sum_{\mu=1}^{4}\sum_{\beta=1}^{4}\left[(\gamma_\mu)_{\alpha\beta}\frac{\partial}{\partial x_\mu} + \left(\frac{mc}{\hbar}\right)\delta_{\alpha\beta}\right]\psi_\beta = 0.$$

Note The fact that the Dirac spinor happens to have four components has nothing to do with our four dimensional space-time; ψ_β does not transform like a four vector under Lorentz transformations.
It is easy to verify that the *gamma-matrices* (*Dirac matrices*) γ_μ satisfy the anticommutation rule

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}.$$

Furthermore, every γ_μ is Hermitian,

$$\gamma_\mu^\dagger = \gamma_\mu,$$

and traceless, i.e.

$$\mathrm{Tr}\,\gamma_\mu = 0.$$

Let's multiply the equation

$$\left(\boldsymbol{\gamma}\cdot\nabla + \gamma_4\frac{\partial}{\partial(ix_0)}\right)\psi + \frac{mc}{\hbar}\psi = 0$$

on both sides by the matrix γ_4 and we get

$$\left(c\hbar\gamma_4\boldsymbol{\gamma}\cdot\nabla - i\hbar\frac{\partial}{\partial t}\right)\psi + \gamma_4 mc^2\psi = 0.$$

Denote

$$\begin{aligned}\beta &= \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\\ \alpha_k &= i\gamma_4\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0\end{pmatrix},\end{aligned}$$

which satisfy the relations

$$\begin{aligned}\{\alpha_k, \beta\} &= 0\\ \beta^2 &= 1\\ \{\alpha_k, \alpha_l\} &= 2\delta_{kl}.\end{aligned}$$

When we now write

$$H = -ic\hbar\boldsymbol{\alpha}\cdot\nabla + \beta mc^2,$$

the Dirac equation takes the familiar form

$$H\psi = i\hbar\frac{\partial\psi}{\partial t}.$$

We define the *adjungated* spinor $\bar\psi$ like:

$$\bar\psi = \psi^\dagger\gamma_4.$$

Explicitely, if ψ is a column vector

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix},$$

then $\psi^\dagger$ and $\bar\psi$ are row vectors

$$\begin{aligned}\psi^\dagger &= (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)\\ \bar\psi &= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*).\end{aligned}$$

Forming the Hermitean conjugate of the Dirac equation

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi = 0$$

we get

$$\frac{\partial}{\partial x_k}\psi^\dagger\gamma_k + \frac{\partial}{\partial x_4^*}\psi^\dagger\gamma_4 + \frac{mc}{\hbar}\psi^\dagger = 0.$$

We multiply this from right by the matrix γ_4 and end up with the *adjungated equation*

$$-\frac{\partial}{\partial x_\mu}\bar\psi\gamma_\mu + \frac{mc}{\hbar}\bar\psi = 0.$$

Here we have used the relations

$$\begin{aligned}\frac{\partial}{\partial x_4^*} &= \frac{\partial}{\partial(ict)^*} = -\frac{\partial}{\partial x_4}\\ \gamma_k\gamma_4 &= -\gamma_4\gamma_k.\end{aligned}$$

Let's multiply the original Dirac equation

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi = 0$$

from left with the adjungated spinor $\bar\psi$ and the adjungated equation

$$-\frac{\partial}{\partial x_\mu}\bar\psi\gamma_\mu + \frac{mc}{\hbar}\bar\psi = 0$$

from right with the spinor ψ and subtract the resulting equations. We then get

$$\frac{\partial}{\partial x_\mu}(\bar\psi\gamma_\mu\psi) = 0.$$

The quantity

$$s_\mu = ic\bar\psi\gamma_\mu\psi = (c\psi^\dagger\boldsymbol{\alpha}\psi, ic\psi^\dagger\psi)$$

thus satisfies a continuity equation. According to Green's theorem we have

$$\int\bar\psi\gamma_4\psi\,d^3x = \int\psi^\dagger\psi\,d^3x = \text{constant},$$

where the constant can be taken to be 1 with a suitable normalization of ψ. Because $\bar\psi\gamma_4\psi = \psi^\dagger\psi$ is positively definite it can be interpreted as a probability density.
Then

$$s_k = ic\bar\psi\gamma_k\psi = c\psi^\dagger\alpha_k\psi$$

can be identified as the density of the probability current.
Note It can be shown that s_μ transforms like a four vector, so the continuity equation is relativistically covariant.
It can be proved that any sets of four matrices γ_μ and γ'_μ satisfying the anticommutation relations

$$\begin{aligned}\{\gamma_\mu, \gamma_\nu\} &= 2\delta_{\mu\nu}\\ \{\gamma'_\mu, \gamma'_\nu\} &= 2\delta_{\mu\nu},\end{aligned}$$

are related to eachother through a similarity transformation with a non-singular 4×4-matrix S:

$$S\gamma_\mu S^{-1} = \gamma'_\mu.$$

With the help of the matrices γ'_μ the original Dirac equation can be written as

$$\left(S^{-1}\gamma'_\mu S\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right) S^{-1}S\psi = 0.$$

Multiplying this from left with the matrix S we get

$$\left(\gamma'_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi' = 0,$$

where

$$\psi' = S\psi.$$

Thus Dirac's equation is independent on the explicit form of the matrices γ_μ; only the anticommutation of the matrices is relevant. If the matrices γ'_μ are Hermitean the transformation matrix S can be taken to be unitary. It is easy to show that then the probability density and current, for example, are independent on the representation:

$$\bar{\psi}'\gamma'_\mu\psi' = \bar{\psi}\gamma_\mu\psi.$$

Vector potential

When the system is subjected to a vector potential

$$A_\mu = (\boldsymbol{A}, iA_0),$$

we make the ordinary substitutions

$$-i\hbar(\partial/\partial x_\mu) \mapsto -i\hbar(\partial/\partial x_\mu) - eA_\mu/c.$$

The Dirac equation takes now the form

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c}A_\mu\right)\gamma_\mu\psi + \frac{mc}{\hbar}\psi = 0.$$

Assuming that A_μ does not depend on time the time dependence of the spinor ψ can be written as

$$\psi = \psi(\boldsymbol{x}, t)|_{t=0}\, e^{-iEt/\hbar}.$$

Let us write now the Dirac equation for the components ψ_A and ψ_B:

$$\begin{aligned}\left[\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)\right]\psi_B &= \frac{1}{c}(E - eA_0 - mc^2)\psi_A \\ -\left[\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)\right]\psi_A &= -\frac{1}{c}(E - eA_0 + mc^2)\psi_B.\end{aligned}$$

With the help of the latter equation we eliminate ψ_B from the upper equation and get

$$\left[\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)\right]\left[\frac{c^2}{E - eA_0 + mc^2}\right]\left[\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)\right]\psi_A = (E - eA_0 - mc^2)\psi_A.$$

Suppose now that

$$E \approx mc^2, \quad |eA_0| \ll mc^2$$

and measure the energy starting from the rest energy:

$$E^{(\mathrm{NR})} = E - mc^2.$$

We expand

$$\begin{aligned}\frac{c^2}{E - eA_0 + mc^2} &= \frac{1}{2m}\left[\frac{2mc^2}{2mc^2 + E^{(\mathrm{NR})} - eA_0}\right] \\ &= \frac{1}{2m}\left[1 - \frac{E^{(\mathrm{NR})} - eA_0}{2mc^2} + \cdots\right].\end{aligned}$$

This can be taken to be the power series in $(v/c)^2$ since

$$E^{(\mathrm{NR})} - eA_0 \approx [\boldsymbol{p} - (e\boldsymbol{A}/c)]^2/2m \approx mv^2/2.$$

Taking into account only the leading term we get

$$\frac{1}{2m}\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)\psi_A = (E^{(\mathrm{NR})} - eA_0)\psi_A.$$

This can be written as

$$\left[\frac{1}{2m}\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)^2 - \frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B} + eA_0\right]\psi_A = E^{(\mathrm{NR})}\psi_A.$$

Up to the zeroth order of $(v/c)^2$ the component ψ_A is thus the two component Schrödinger-Pauli wave function (multiplied with the factor e^{-imc^2t}) familiar from the non-relativistic quantum mechanics. The equation

$$-\left[\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)\right]\psi_A = -\frac{1}{c}(E - eA_0 + mc^2)\psi_B$$

tells us that the component ψ_B is roughly by the factor

$$|\boldsymbol{p} - (e\boldsymbol{A}/c)|/2mc \approx v/2c$$

"less" than ψ_A. Due to this, provided that $E \approx mc^2$, ψ_A is known as the *big* and ψ_B as the *small* component of the Dirac wave function ψ.

We obtain relativistic corrections only when we consider the second or higher order terms of the expansion

$$\begin{aligned}\frac{c^2}{E - eA_0 + mc^2} &= \frac{1}{2m}\left[\frac{2mc^2}{2mc^2 + E^{(\mathrm{NR})} - eA_0}\right] \\ &= \frac{1}{2m}\left[1 - \frac{E^{(\mathrm{NR})} - eA_0}{2mc^2} + \cdots\right].\end{aligned}$$

Let us suppose now that

$$\boldsymbol{A} = 0.$$

The wave equation is then

$$H_A\psi_A = E^{(\mathrm{NR})}\psi_A,$$

where

$$H_A = (\boldsymbol{\sigma}\cdot\boldsymbol{p})\frac{1}{2m}\left(1 - \frac{E^{(\mathrm{NR})} - eA_0}{2mc^2}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{p}) + eA_0.$$

This wave equation looks like a time independent Schrödinger equation for the wave function ψ_A. However,

- evaluating corrections up to the order $(v/c)^2$ the wave function ψ_A is not normalized because the probability interpretation of Dirac's theory requires that
$$\int (\psi_A^\dagger \psi_A + \psi_B^\dagger \psi_B)\, d^3x = 1,$$
where ψ_B already of the order v/c.
- explicitely writing down the expression for the operator H_A we see that it contains the non-Hermitian term $i\hbar \boldsymbol{E}\cdot\boldsymbol{p}$.
- the equation is not an eigenvalue equation since H_A itself contains the term $E^{(\mathrm{NR})}$.

Up to the order $(v/c)^2$ the normalization condition can now be written as
$$\int \psi_A^\dagger \left(1+\frac{\boldsymbol{p}^2}{4m^2c^2}\right)\psi_A\, d^3x \approx 1,$$
because according to the equation
$$-\left[\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{e\boldsymbol{A}}{c}\right)\right]\psi_A = -\frac{1}{c}(E-eA_0+mc^2)\psi_B$$
we have
$$\psi_B \approx \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2mc}\psi_A.$$
It is worthwhile to define the new two component wave function Ψ:
$$\Psi = \Omega\psi_A,$$
where
$$\Omega = 1+\frac{\boldsymbol{p}^2}{8m^2c^2}.$$
Now Ψ is up to the order $(v/c)^2$ normalized correctly because
$$\int \Psi^\dagger\Psi\, d^3x \approx \int \psi_A^\dagger\left(1+\frac{\boldsymbol{p}^2}{4m^2c^2}\right)\psi_A\, d^3x.$$
We multiply the equation
$$H_A\psi_A = E^{(\mathrm{NR})}\psi_A,$$
on both sides with the operator
$$\Omega^{-1} = 1-(\boldsymbol{p}^2/8m^2c^2),$$
and get
$$\Omega^{-1}H_A\Omega^{-1}\Psi = E^{(\mathrm{NR})}\Omega^{-2}\Psi.$$
Explicitely, up to the order $(v/c)^2$ this can be written as
$$\left[\frac{\boldsymbol{p}^2}{2m}+eA_0-\left\{\frac{\boldsymbol{p}^2}{8m^2c^2},\left(\frac{\boldsymbol{p}^2}{2m}+eA_0\right)\right\}\right.$$
$$\left.-\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{2m}\left(\frac{E^{(\mathrm{NR})}-eA_0}{2mc^2}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{p})\right]\Psi$$
$$= E^{(\mathrm{NR})}\left(1-\frac{\boldsymbol{p}^2}{4m^2c^2}\right)\Psi.$$

Writing $E^{(\mathrm{NR})}\boldsymbol{p}^2$ in the form $\frac{1}{2}\{E^{(\mathrm{NR})},\boldsymbol{p}^2\}$ we get
$$\left[\frac{\boldsymbol{p}^2}{2m}+eA_0-\frac{\boldsymbol{p}^4}{8m^3c^2}\right.$$
$$+\frac{1}{8m^2c^2}\Big(\{\boldsymbol{p}^2,(E^{(\mathrm{NR})}-eA_0)\}$$
$$\left.-2(\boldsymbol{\sigma}\cdot\boldsymbol{p})(E^{(\mathrm{NR})}-eA_0)(\boldsymbol{\sigma}\cdot\boldsymbol{p})\Big)\right]\Psi$$
$$= E^{(\mathrm{NR})}\Psi.$$
Because for arbitrary operators A and B
$$\{A^2,B\}-2ABA = [A,[A,B]]$$
holds we can, by setting
$$\begin{aligned}\boldsymbol{\sigma}\cdot\boldsymbol{p} &= A\\ E^{(\mathrm{NR})}-eA_0 &= B,\end{aligned}$$
reduce the equation into the form
$$\left[\frac{\boldsymbol{p}^2}{2m}+eA_0-\frac{\boldsymbol{p}^4}{8m^3c^2}\right.$$
$$\left.-\frac{e\hbar\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p})}{4m^2c^2}-\frac{e\hbar^2}{8m^2c^2}\nabla\cdot\boldsymbol{E}\right]\Psi$$
$$= E^{(\mathrm{NR})}\Psi.$$
In the derivation of the equation we have employed the relations
$$\begin{aligned}[\boldsymbol{\sigma}\cdot\boldsymbol{p},(E^{(\mathrm{NR})}-eA_0)] &= -ie\hbar\boldsymbol{\sigma}\cdot\boldsymbol{E}\\ [\boldsymbol{\sigma}\cdot\boldsymbol{p},-ie\hbar\boldsymbol{\sigma}\cdot\boldsymbol{E}] &= -e\hbar^2\nabla\cdot\boldsymbol{E}\\ &\quad -2e\hbar\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p}),\end{aligned}$$
the validity of which can be verified by noting that
$$\begin{aligned}\nabla A_0 &= -\boldsymbol{E}\\ \nabla\times\boldsymbol{E} &= 0.\end{aligned}$$
The resulting equation is a proper Schrödinger equation for a two component wave function.
Physical interpretation
We look at the meaning of the terms in the equation
$$\left[\frac{\boldsymbol{p}^2}{2m}+eA_0-\frac{\boldsymbol{p}^4}{8m^3c^2}\right.$$
$$\left.-\frac{e\hbar\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p})}{4m^2c^2}-\frac{e\hbar^2}{8m^2c^2}\nabla\cdot\boldsymbol{E}\right]\Psi$$
$$= E^{(\mathrm{NR})}\Psi.$$

1. The term $\frac{\boldsymbol{p}^2}{2m}+eA_0$ gives the non-realtivistic Schrödinger equation.
2. The term $-\frac{\boldsymbol{p}^4}{8m^3c^2}$ is a relativistic correction to the kinetic energy as can be seen from the expansion
$$\sqrt{(mc^2)^2+|\boldsymbol{p}|^2c^2}-mc^2 = \frac{|\boldsymbol{p}|^2}{2m}-\frac{|\boldsymbol{p}|^4}{8m^3c^2}+\cdots.$$

3. The term $-\frac{e\hbar\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p})}{4m^2c^2}$ describes the interaction between the spin of a moving electron and electric field. Intuitively this, so called *Thomas term*, is due to the fact that the moving electron experiences an apparent magnetic field $\boldsymbol{E}\times(\boldsymbol{v}/c)$. If the electric field is a central field,

$$eA_0 = V(r),$$

it can be written in the form

$$\begin{aligned} -\frac{e\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p}) &= -\frac{\hbar}{4m^2c^2}\left(-\frac{1}{r}\frac{dV}{dr}\right)\boldsymbol{\sigma}\cdot(\boldsymbol{x}\times\boldsymbol{p}) \\ &= \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\boldsymbol{S}\cdot\boldsymbol{L}, \end{aligned}$$

where we have substituted

$$\boldsymbol{S} = \hbar\boldsymbol{\sigma}/2.$$

So we actually have a spin orbit interaction.

4. The term $-\frac{e\hbar^2}{8m^2c^2}\nabla\cdot\boldsymbol{E}$ is known as the *Darwin term*. Its meaning can be deduced when we note that $\nabla\cdot\boldsymbol{E}$ is the charge density. For example, in the hydrogen atom where $\nabla\cdot\boldsymbol{E} = -e\delta(\boldsymbol{x})$ it causes the energy shift

$$\int \frac{e^2\hbar^2}{8m^2c^2}\delta(\boldsymbol{x})|\psi^{(\text{Schrö})}|^2\, d^3x = \frac{e^2\hbar^2}{8m^2c^2}|\psi^{(\text{Schrö})}|^2\Big|_{\boldsymbol{x}=0},$$

which differs from zero only in the s-state.