SU(2)

In the two dimensional space

$$\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$$

the spin operators

$$\begin{aligned}
S_x &= \left(\frac{\hbar}{2}\right)\{(|S_z;\uparrow\rangle\langle S_z;\downarrow|) + (|S_z;\downarrow\rangle\langle S_z;\uparrow|)\} \\
S_y &= \left(\frac{i\hbar}{2}\right)\{-(|S_z;\uparrow\rangle\langle S_z;\downarrow|) + (|S_z;\downarrow\rangle\langle S_z;\uparrow|)\} \\
S_z &= \left(\frac{\hbar}{2}\right)\{(|S_z;\uparrow\rangle\langle S_z;\uparrow|) - (|S_z;\downarrow\rangle\langle S_z;\downarrow|)\}
\end{aligned}$$

satisfy the angular momentum commutation relations

$$[S_x, S_y] = i\hbar S_z + \text{cyclic permutations.}$$

Thus the smallest dimension where these commutation relations can be realized is 2.
The state

$$|\alpha\rangle = |S_z;\uparrow\rangle\langle S_z;\uparrow|\alpha\rangle + |S_z;\downarrow\rangle\langle S_z;\downarrow|\alpha\rangle$$

behaves in the rotation

$$\mathcal{D}_z(\phi) = \exp\left(-\frac{iS_z\phi}{\hbar}\right)$$

like

$$\begin{aligned}
\mathcal{D}_z(\phi)|\alpha\rangle &= \exp\left(-\frac{iS_z\phi}{\hbar}\right)|\alpha\rangle \\
&= e^{-i\phi/2}|S_z;\uparrow\rangle\langle S_z;\uparrow|\alpha\rangle \\
&\quad + e^{i\phi/2}|S_z;\downarrow\rangle\langle S_z;\downarrow|\alpha\rangle.
\end{aligned}$$

In particular:

$$\mathcal{D}_z(2\pi)|\alpha\rangle = -|\alpha\rangle$$

and one needs

$$\mathcal{D}_z(4\pi)|\alpha\rangle = |\alpha\rangle.$$

Notice however, that the minus sign disappears in the expectation value.

Spin precession revisited

When the Hamiltonian is

$$H = \omega_c S_z$$

the time evolution operator is

$$\mathcal{U}(t,0) = \exp\left(-\frac{iS_z\omega_c t}{\hbar}\right) = \mathcal{D}_z(\omega_c t).$$

Looking at the equations

$$\begin{aligned}
\langle J_x \rangle &\longrightarrow_R \langle J_x \rangle \cos\phi - \langle J_y \rangle \sin\phi \\
\langle J_y \rangle &\longrightarrow_R \langle J_y \rangle \cos\phi + \langle J_x \rangle \sin\phi \\
\langle J_z \rangle &\longrightarrow_R \langle J_z \rangle
\end{aligned}$$

we see that in this case the time-evolution operator acts like rotation operator and thus

$$\begin{aligned}
\langle S_x \rangle_t &= \langle S_x \rangle_{t=0} \cos\omega_c t - \langle S_y \rangle_{t=0} \sin\omega_c t \\
\langle S_y \rangle_t &= \langle S_y \rangle_{t=0} \cos\omega_c t + \langle S_x \rangle_{t=0} \sin\omega_c t \\
\langle S_z \rangle_t &= \langle S_z \rangle_{t=0}.
\end{aligned}$$

We see that

- the spin returns to its original direction after time $t = 2\pi/\omega_c$.

By looking at the time-evolution of the state ket itself

$$\begin{aligned}
|\alpha, t_0\rangle &= |S_z;\uparrow\rangle\langle S_z;\uparrow|\alpha, t_0\rangle + |S_z;\downarrow\rangle\langle S_z;\downarrow|\alpha, t_0\rangle \\
|\alpha, t\rangle &= e^{-iwt/2}|S_z;\uparrow\rangle\langle S_z;\uparrow|\alpha, t_0\rangle \\
&+ e^{+iwt/2}|S_z;\downarrow\rangle\langle S_z;\downarrow|\alpha, t_0\rangle
\end{aligned}$$

we see that

- the wave vector returns to its original value after time $t = 4\pi/\omega_c$.

Indeed, this has been verified experimentally

Matrix representation
In the basis $\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$ the base vectors are represented as

$$\begin{aligned}
|S_z;\uparrow\rangle &\mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_\uparrow \qquad |S_z;\downarrow\rangle \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_\downarrow \\
\langle S_z;\uparrow| &\mapsto (1,0) \equiv \chi_\uparrow^\dagger \qquad \langle S_z;\downarrow| \mapsto (0,1) \equiv \chi_\downarrow^\dagger,
\end{aligned}$$

so an arbitrary state vector is represented as

$$\begin{aligned}
|\alpha\rangle &\mapsto \begin{pmatrix} \langle S_z;\uparrow|\alpha\rangle \\ \langle S_z;\downarrow|\alpha\rangle \end{pmatrix} \\
\langle\alpha| &\mapsto (\langle\alpha|S_z;\uparrow\rangle, \langle\alpha|S_z;\downarrow\rangle).
\end{aligned}$$

The column vector
$$\chi = \begin{pmatrix} \langle S_z;\uparrow|\alpha\rangle \\ \langle S_z;\downarrow|\alpha\rangle \end{pmatrix} \equiv \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}$$

is called the *two component spinor*
Quite obviously, $\chi^\dagger = \left(c_\uparrow^*, c_\downarrow^*\right)$

Pauli's spin matrices
Pauli's spin matrices σ_i are defined via the relations

$$(S_k)_{ij} \equiv \left(\frac{\hbar}{2}\right)(\sigma_k)_{ij},$$

where the matrix elements are evaluated in the basis $\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$.
For example

$$S_1 = S_x = \left(\frac{\hbar}{2}\right)\{(|S_z;\uparrow\rangle\langle S_z;\downarrow|) + (|S_z;\downarrow\rangle\langle S_z;\uparrow|)\},$$

so

$$\begin{aligned} (S_1)_{11} = (S_1)_{22} &= 0 \\ (S_1)_{12} = (S_1)_{21} &= \frac{\hbar}{2}, \end{aligned}$$

or

$$(S_1) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus we get

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spin matrices satisfy the anticommutation relations

$$\{\sigma_i, \sigma_j\} \equiv \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$$

and the commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

Moreover, we see that

$$\begin{aligned} \sigma_i^\dagger &= \sigma_i, \\ \det(\sigma_i) &= -1, \\ \mathrm{tr}(\sigma_i) &= 0. \end{aligned}$$

Often the collective vector notation

$$\boldsymbol{\sigma} \equiv \sigma_1\hat{\boldsymbol{x}} + \sigma_2\hat{\boldsymbol{y}} + \sigma_3\hat{\boldsymbol{z}}.$$

is used for spin matrices. For example we get

$$\begin{aligned} \boldsymbol{\sigma}\cdot\boldsymbol{a} &\equiv \sum_k a_k\sigma_k \\ &= \begin{pmatrix} +a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}. \end{aligned}$$

and

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) &= \sum_{j,k}\sigma_j a_j \sigma_k b_k \\
&= \sum_{j,k}\frac{1}{2}\left(\{\sigma_j,\sigma_k\}+[\sigma_j,\sigma_k]\right)a_j b_k \\
&= \sum_{j,k}(\delta_{jk}+i\epsilon_{jki}\sigma_i)a_j b_k \\
&= \boldsymbol{a}\cdot\boldsymbol{b}+i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b}).
\end{aligned}
$$

A special case ($\boldsymbol{a}$ real) of the latter formula is

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})^2 = |\boldsymbol{a}|^2.$$

Now

$$
\mathcal{D}(\hat{\boldsymbol{n}},\phi) = \exp\left(-\frac{i\boldsymbol{S}\cdot\hat{\boldsymbol{n}}\phi}{\hbar}\right) \mapsto \exp\left(-\frac{i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi}{2}\right) =
$$

$$
\mathbf{1}\cos\left(\frac{\phi}{2}\right) - i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\sin\left(\frac{\phi}{2}\right) =
$$

$$
\begin{pmatrix}
\cos\left(\frac{\phi}{2}\right) - in_z\sin\left(\frac{\phi}{2}\right) & (-in_x-n_y)\sin\left(\frac{\phi}{2}\right) \\
(-in_x+n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z\sin\left(\frac{\phi}{2}\right)
\end{pmatrix}
$$

and the spinors behave in rotations like

$$\chi \longrightarrow \exp\left(-\frac{i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi}{2}\right)\chi.$$

The first step is realized when Taylor expanding the exponential function and noticing that

$$
(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})^k = \{ \begin{array}{cc} 1 & k \text{ even} \\ \boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} & k \text{ odd} \end{array}
$$

Note the notation $\boldsymbol{\sigma}$ *does not* mean that $\boldsymbol{\sigma}$ would behave in rotations like a vector, $\sigma_k \longrightarrow {}_R\sigma_k$. Instead, quantity

$$\chi^\dagger\sigma_k\chi \longrightarrow \sum_l R_{kl}\chi^\dagger\sigma_l\chi.$$

transforms like a vector.
For all directions $\hat{\boldsymbol{n}}$ one has

$$
\left.\exp\left(-\frac{i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi}{2}\right)\right|_{\phi=2\pi} = -\mathbf{1}, \quad \text{for any } \hat{\boldsymbol{n}}.
$$

Euler's angles
The spinor rotation matrices corresponding to rotations around z and y axes are

$$\begin{aligned}
\mathcal{D}_z(\alpha) = \exp(-i\sigma_3\alpha/2) &\quad\mapsto\quad \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \\
\mathcal{D}_y(\beta) = \exp(-i\sigma_2\beta/2) &\quad\mapsto\quad \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix}.
\end{aligned}$$

With the help of Euler's angles α, β and γ the rotation matrices can be written as

$$\mathcal{D}(\alpha,\beta,\gamma) \mapsto \mathcal{D}^{(\frac{1}{2})}(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) & -e^{-i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) \\ e^{i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) & e^{i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) \end{pmatrix}.$$

We seek for the eigenspinor of the matrix $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$:

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\chi = \chi.$$

Now

$$\hat{\boldsymbol{n}} = \begin{pmatrix} \sin\beta\cos\alpha \\ \sin\beta\sin\alpha \\ \cos\beta \end{pmatrix},$$

so

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} = \begin{pmatrix} \cos\beta & \sin\beta e^{-i\alpha} \\ \sin\beta e^{i\alpha} & -\cos\beta \end{pmatrix}.$$

The state where the spin is parallel to the unit vector $\hat{\boldsymbol{n}}$, is obviously invariant under rotations

$$\mathcal{D}_{\hat{\boldsymbol{n}}}(\phi) = e^{-i\boldsymbol{S}\cdot\hat{\boldsymbol{n}}/\hbar}$$

and thus an eigenstate of the operator $\boldsymbol{S}\cdot\hat{\boldsymbol{n}}$.
This kind of state can be obtained by rotating the state $|S_z;\uparrow\rangle$

1. angle β around y axis,
2. angle α around z axis,

i.e.

$$\begin{aligned}
\boldsymbol{S}\cdot\hat{\boldsymbol{n}}|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};\uparrow\rangle &= \boldsymbol{S}\cdot\hat{\boldsymbol{n}}\mathcal{D}(\alpha,\beta,0)|S_z;\uparrow\rangle \\
&= \left(\frac{\hbar}{2}\right)\mathcal{D}(\alpha,\beta,0)|S_z;\uparrow\rangle \\
&= \left(\frac{\hbar}{2}\right)|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};\uparrow\rangle.
\end{aligned}$$

Correspondingly for spinors the vector

$$\chi = \mathcal{D}^{(\frac{1}{2})}(\alpha, \beta, 0)|S_z; \uparrow\rangle = \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) e^{-i\alpha/2} \\ \sin\left(\frac{\beta}{2}\right) e^{i\alpha/2} \end{pmatrix}$$

is an eigenstate of the matrix $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}$.

SU(2)
As a representation of rotations the 2×2-matrices

$$\mathcal{D}^{(\frac{1}{2})}(\hat{\boldsymbol{n}}, \phi) = e^{-i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi/2}$$

form obviously a group. These matrices have two characteristic properties:

1. unitarity
$$\left(\mathcal{D}^{(\frac{1}{2})}\right)^{\dagger} = \left(\mathcal{D}^{(\frac{1}{2})}\right)^{-1},$$

2. unimodularity
$$\left|\mathcal{D}^{(\frac{1}{2})}\right| = 1.$$

A unitary unimodular matrix can be written as

$$U(a, b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}.$$

The unimodularity condition gives

$$1 = |U| = |a|^2 + |b|^2,$$

and we are left with 3 free parameters.
The unitarity condition is automatically satisfied because

$$\begin{aligned} U(a,b)^{\dagger}U(a,b) &= \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \\ &= \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix} = 1. \end{aligned}$$

Matrices $U(a, b)$ form a group since

- the matrix
$$U(a_1, b_1)U(a_2, b_2) = U(a_1a_2 - b_1b_2^*, a_1b_2 + a_2^*b_1)$$
is unimodular because
$$\begin{aligned} &|U(a_1a_2 - b_1b_2^*, a_1b_2 + a_2^*b_1)| = \\ &\qquad |a_1a_2 - b_1b_2^*|^2 + |a_1b_2 + a_2^*b_1|^2 = 1, \end{aligned}$$
and thus also unitary.

- as a unitary matrix U has the inverse matrix:

$$U^{-1}(a,b) = U^{\dagger}(a,b) = U(a^*, -b).$$

- the unit matrix 1 is unitary and unimodular.

The group is called SU(2).
Comparing with the previous spinor representation

$$\mathcal{D}^{(\frac{1}{2})}(\hat{\boldsymbol{n}}, \phi) = \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) - in_z \sin\left(\frac{\phi}{2}\right) & (-in_x - n_y)\sin\left(\frac{\phi}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z \sin\left(\frac{\phi}{2}\right) \end{pmatrix}$$

we see that

$$\begin{aligned} \mathrm{Re}(a) &= \cos\left(\frac{\phi}{2}\right) & \mathrm{Im}(a) &= -n_z \sin\left(\frac{\phi}{2}\right) \\ \mathrm{Re}(b) &= -n_y \sin\left(\frac{\phi}{2}\right) & \mathrm{Im}(b) &= -n_x \sin\left(\frac{\phi}{2}\right). \end{aligned}$$

The complex numbers a and b are known as *Cayley-Klein's parameters.*
Note O(3) and SU(2) are *not* isomorphic.
Example
In O(3): 2π- and 4π-rotations $\mapsto 1$
In SU(2): 2π-rotation $\mapsto -1$ and 4π-rotation $\mapsto 1$.
The operations $U(a,b)$ and $U(-a,-b)$ in SU(2) correspond to a single matrix of O(3). The map SU(2) $\mapsto$ O(3) is thus 2 to 1. The groups are, however, locally isomorphic.