Angular momentum algebra

It is easy to see that the operator

$$\boldsymbol{J}^2 = J_x J_x + J_y J_y + J_z J_z$$

commutes with the operators J_x, J_y and J_z,

$$[\boldsymbol{J}^2, J_i] = 0.$$

Since J_i, J_k do not commute, we can choose only one of them to be observable together with $\boldsymbol{J}^2$
We choose the component J_z and denote the common eigenstate of the operators $\boldsymbol{J}^2$ and J_z by $|j,m\rangle$, i.e. so that

$$\begin{aligned} \boldsymbol{J}^2|j,m\rangle &= a_j|j,m\rangle \\ J_z|j,m\rangle &= b_m|j,m\rangle \end{aligned}$$

We know (QM II) that

$$\begin{aligned} \boldsymbol{J}^2|j,m\rangle &= j(j+1)\hbar^2|j,m\rangle,\ j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \\ J_z|j,m\rangle &= m\hbar|j,m\rangle,\ m = -j, -j+1, \ldots, j-1, j. \end{aligned}$$

Above equations are derived easily by repeatedly applying the ladder operators.
Notice,
if j integer $\rightarrow m$ integer,
if j half-integer $\rightarrow m$ half-integer.
Also notice, the quantization of angular momentum arises solely from the commutation relations, which follow from the properties of rotations combined with the definition of J_k as the generator of rotation.
We define the ladder operators J_+ and J_-:

$$J_\pm \equiv J_x \pm iJ_y.$$

They satisfy the commutation relations

$$\begin{aligned} [J_+, J_-] &= [J_x \pm iJ_y, J_x \mp iJ_y] = 2\hbar J_z \\ [J_z, J_\pm] &= [J_z, J_x \pm iJ_y] \qquad = \pm\hbar J_\pm \\ \left[\boldsymbol{J}^2, J_\pm\right] &= \left[\boldsymbol{J}^2, J_x \pm iJ_y\right] \qquad = 0. \end{aligned}$$

All of which follow from the

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$$

We see that

$$\begin{aligned} J_z\Bigg(J_\pm|j,m\rangle\Bigg) &= ([J_z, J_\pm] + J_\pm J_z)|j,m\rangle \\ &= (m\pm 1)\hbar\Bigg(J_\pm|j,m\rangle\Bigg) \end{aligned}$$

and so $J_\pm$ raise/lower the eigenvalue by a quantum unit (compare with harmonic oscillator ladder operators)
In contrary, $J_\pm$ does not change the eigenvalue of $\boldsymbol{J}^2$

$$\boldsymbol{J}^2 J_\pm|j,m\rangle = J_\pm \boldsymbol{J}^2|j,m\rangle = j(j+1)\hbar^2 J_\pm|j,m\rangle,$$

thus, we must have

$$J_\pm|j,m\rangle = c^\pm_{j,m}|j,m\pm 1\rangle$$

The factor $c^\pm_{j,m}$ can be deduced from the normalization condition

$$\langle j,m|j',m'\rangle = \delta_{jj'}\delta_{mm'}.$$

Considering

$$\begin{aligned}
\langle j,m|J_+^\dagger J_+|j,m\rangle &= \langle j,m|\boldsymbol{J}^2 - J_z^2 - \hbar J_z|j,m\rangle \\
&= \hbar^2\left[j(j+1) - m^2 - m\right] \\
&= \hbar^2\left[j(j+1) - m(m+1)\right] \\
&= \hbar^2(j-m)(j+m+1) \\
&= |c^+_{j,m}|^2
\end{aligned}$$

$c^-_{j,m}$ is found similarly and we end up with

$$J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\hbar|j,m\pm 1\rangle.$$

Matrix elements will be

$$\begin{aligned}
\langle j',m'|\boldsymbol{J}^2|j,m\rangle &= j(j+1)\hbar^2\delta_{j'j}\delta_{m'm} \\
\langle j',m'|J_z|j,m\rangle &= m\hbar\delta_{j'j}\delta_{m'm} \\
\langle j',m'|J_\pm|j,m\rangle &= \sqrt{(j\mp m)(j\pm m+1)}\hbar\delta_{j'j}\delta_{m',m\pm 1}.
\end{aligned}$$

Representation of the Rotation operator
We define *Wigner's function*:

$$\mathcal{D}^{(j)}_{m'm}(R) = \langle j,m'|\exp\left(-\frac{i\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\phi}{\hbar}\right)|j,m\rangle.$$

Since

$$[\boldsymbol{J}^2,\mathcal{D}(R)] = [\boldsymbol{J}^2,\exp\left(-\frac{i\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\phi}{\hbar}\right)] = 0,$$

$$\mathcal{D}^{(j)}_{m'm}(R)|j,m\rangle$$

is an eigenket of $\boldsymbol{J}^2$ with the same eigenvalue as the unrotated ket.
Thus, we see that $\mathcal{D}(R)$ does not chance the j-quantum number, so it cannot have non zero matrix elements between states with different j values.
The matrix with matrix elements $\mathcal{D}^{(j)}_{m'm}(R)$ is the $(2j+1)$-dimensional irreducible representation of a general rotation operator $\mathcal{D}(R)$ in the ket space.
The matrices $\mathcal{D}^{(j)}_{m'm}(R)$ form a group:

- The product of matrices belongs to the group:

$$\mathcal{D}^{(j)}_{m''m}(R_1R_2) = \sum_{m'} \mathcal{D}^{(j)}_{m''m'}(R_1)\mathcal{D}^{(j)}_{m'm}(R_2),$$

  where R_1R_2 is the combined rotation of the rotations R_1 and R_2,

- the inverse operation belongs to the group:

$$\mathcal{D}^{(j)}_{m'm}(R^{-1}) = \mathcal{D}^{(j)^*}_{mm'}(R).$$

The state vectors $|j,m\rangle$ transform in rotations like

$$\begin{aligned}\mathcal{D}(R)|j,m\rangle &= \sum_{m'} |j,m'\rangle\langle j,m'|\mathcal{D}(R)|j,m\rangle \\ &= \sum_{m'} |j,m'\rangle \mathcal{D}^{(j)}_{m'm}(R).\end{aligned}$$

With the help of the Euler angles

$$\begin{aligned}&\mathcal{D}^{(j)}_{m'm}(R) = \\ &\quad \langle j,m'|\exp\left(-\frac{iJ_z\alpha}{\hbar}\right)\exp\left(-\frac{iJ_y\beta}{\hbar}\right)\exp\left(-\frac{iJ_z\gamma}{\hbar}\right)|j,m\rangle \\ &\quad = e^{-i(m'\alpha+m\gamma)}d^{(j)}_{m'm}(\beta),\end{aligned}$$

where

$$d^{(j)}_{m'm}(\beta) \equiv \langle j,m'|\exp\left(-\frac{iJ_y\beta}{\hbar}\right)|j,m\rangle.$$

Functions $d^{(j)}_{m'm}$ can be evaluated using *Wigner's formula*

$$\begin{aligned}&d^{(j)}_{m'm}(\beta) = \\ &\quad \sum_k (-1)^{k-m+m'} \\ &\quad \times \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-k)!k!(j-k-m')!(k-m+m')!} \\ &\quad \times \left(\cos\frac{\beta}{2}\right)^{2j-2k+m-m'} \times \left(\sin\frac{\beta}{2}\right)^{2k-m+m'}.\end{aligned}$$

Orbital angular momentum
The components of the classically analogous operator $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$ satisfy the commutation relations

$$[L_i, L_j] = i\epsilon_{ijk}\hbar L_k.$$

Using the spherical coordinates to label the position eigenstates,

$$|\boldsymbol{x}'\rangle = |r,\theta,\phi\rangle,$$

one can show that

$$\begin{aligned}
\langle \boldsymbol{x}'|L_z|\alpha\rangle &= -i\hbar\frac{\partial}{\partial\phi}\langle \boldsymbol{x}'|\alpha\rangle \\
\langle \boldsymbol{x}'|L_x|\alpha\rangle &= -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta}-\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\langle \boldsymbol{x}'|\alpha\rangle \\
\langle \boldsymbol{x}'|L_y|\alpha\rangle &= -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\langle \boldsymbol{x}'|\alpha\rangle \\
\langle \boldsymbol{x}'|L_\pm|\alpha\rangle &= -i\hbar e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\phi}\right)\langle \boldsymbol{x}'|\alpha\rangle \\
\langle \boldsymbol{x}'|\boldsymbol{L}^2|\alpha\rangle &= -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] \\
&\quad \times\langle \boldsymbol{x}'|\alpha\rangle.
\end{aligned}$$

We denote the common eigenstate of the operators $\boldsymbol{L}^2$ and L_z by the ket-vector $|l,m\rangle$, i.e.

$$\begin{aligned}
L_z|l,m\rangle &= m\hbar|l,m\rangle \\
\boldsymbol{L}^2|l,m\rangle &= l(l+1)\hbar^2|l,m\rangle.
\end{aligned}$$

Since $\mathcal{R}^3$ can be represented as the direct product

$$\mathcal{R}^3 = \mathcal{R}\times\Omega,$$

where Ω is the surface of the unit sphere (position=distance from the origin and direction) the position eigenstates can be written correspondingly as

$$|\boldsymbol{x}'\rangle = |r\rangle|\hat{\boldsymbol{n}}\rangle.$$

Here the state vectors $|\hat{\boldsymbol{n}}\rangle$ form a complete basis on the surface of the sphere, i.e.

$$\int d\Omega_{\hat{\boldsymbol{n}}}\,|\hat{\boldsymbol{n}}\rangle\langle\hat{\boldsymbol{n}}| = 1.$$

We define the spherical harmonic function:

$$Y_l^m(\theta,\phi) = Y_l^m(\hat{\boldsymbol{n}}) = \langle\hat{\boldsymbol{n}}|l,m\rangle.$$

The scalar product of the vector $\langle\hat{\boldsymbol{n}}|$ with the equations

$$\begin{aligned}
L_z|l,m\rangle &= m\hbar|l,m\rangle \\
\boldsymbol{L}^2|l,m\rangle &= l(l+1)\hbar^2|l,m\rangle
\end{aligned}$$

gives

$$-i\hbar\frac{\partial}{\partial\phi}Y_l^m(\theta,\phi) = m\hbar Y_l^m(\theta,\phi)$$

and

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+l(l+1)\right]Y_l^m = 0.$$

Y_l^m and $\mathcal{D}^{(l)}$
The state

$$|\hat{\boldsymbol{n}}\rangle = |\theta, \phi\rangle$$

is obtained from the state $|\hat{\boldsymbol{z}}\rangle$ rotating it first by the angle θ around y-axis and then by the angle ϕ around z-axis:

$$\begin{aligned} |\hat{\boldsymbol{n}}\rangle &= \mathcal{D}(R)|\hat{\boldsymbol{z}}\rangle \\ &= \mathcal{D}(\alpha = \phi, \beta = \theta, \gamma = 0)|\hat{\boldsymbol{z}}\rangle \\ &= \sum_{l,m} \mathcal{D}(\phi, \theta, 0)|l, m\rangle\langle l, m|\hat{\boldsymbol{z}}\rangle. \end{aligned}$$

Furthermore

$$\langle l, m|\hat{\boldsymbol{n}}\rangle = Y_l^{m*}(\theta, \phi) = \sum_m \mathcal{D}^{(l)}_{m'm}(\phi, \theta, 0)\langle l, m|\hat{\boldsymbol{z}}\rangle.$$

Now

$$\langle l, m|\hat{\boldsymbol{z}}\rangle = Y_l^{m*}(0, \phi) = \sqrt{\frac{(2l+1)}{4\pi}}\delta_{m0},$$

so

$$Y_l^{m*}(\theta, \phi) = \sqrt{\frac{(2l+1)}{4\pi}}\mathcal{D}^{(l)}_{m0}(\phi, \theta, \gamma = 0)$$

or

$$\mathcal{D}^{(l)}_{m0}(\alpha, \beta, 0) = \sqrt{\frac{4\pi}{(2l+1)}}Y_l^{m*}(\theta, \phi)\bigg|_{\beta,\alpha}.$$

As a special case

$$\mathcal{D}^{(l)}_{00}(\theta, \phi, 0) = d^{(l)}_{00}(\theta) = P_l(\cos\theta).$$

Coupling of angular momenta
We consider two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. If now A_i is an operator in the space $\mathcal{H}_i$, the notation $A_1 \otimes A_2$ means the operator

$$A_1 \otimes A_2|\alpha\rangle_1 \otimes |\beta\rangle_2 = (A_1|\alpha\rangle_1) \otimes (A_2|\beta\rangle_2)$$

in the product space. Here $|\alpha\rangle_i \in \mathcal{H}_i$. In particular,

$$A_1 \otimes 1_2|\alpha\rangle_1 \otimes |\beta\rangle_2 = (A_1|\alpha\rangle_1) \otimes |\beta\rangle_2,$$

where 1_i is the identity operator of the space $\mathcal{H}_i$. Correspondingly $1_1 \otimes A_2$ operates only in the subspace $\mathcal{H}_2$ of the product space. Usually the subspace of the identity operators, or even the identity operator itself, is not shown, for example

$$A_1 \otimes 1_2 = A_1 \otimes 1 = A_1.$$

It is easy to verify that operators operating in different subspace commute, i.e.

$$[A_1 \otimes 1_2, 1_1 \otimes A_2] = [A_1, A_2] = 0.$$

In particular we consider two angular momenta $\boldsymbol{J_1}$ and $\boldsymbol{J_2}$ operating in two different Hilbert spaces. They commute:

$$[J_{1i}, J_{2j}] = 0.$$

The infinitesimal rotation affecting both Hilbert spaces is

$$\left(1 - \frac{i\boldsymbol{J}_1 \cdot \hat{\boldsymbol{n}}\delta\phi}{\hbar}\right) \otimes \left(1 - \frac{i\boldsymbol{J}_2 \cdot \hat{\boldsymbol{n}}\delta\phi}{\hbar}\right) =$$
$$1 - \frac{i(\boldsymbol{J}_1 \otimes 1 + 1 \otimes \boldsymbol{J}_2) \cdot \hat{\boldsymbol{n}}\delta\phi}{\hbar}.$$

The components of the total angular momentum

$$\boldsymbol{J} = \boldsymbol{J}_1 \otimes 1 + 1 \otimes \boldsymbol{J}_2 = \boldsymbol{J}_1 + \boldsymbol{J}_2$$

obey the commutation relations

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k,$$

i.e. $\boldsymbol{J}$ *is angular momentum.*
A finite rotation is constructed analogously:

$$\mathcal{D}_1(R) \otimes \mathcal{D}_2(R) = \exp\left(-\frac{\boldsymbol{J}_1 \cdot \hat{\boldsymbol{n}}\phi}{\hbar}\right) \otimes \exp\left(-\frac{\boldsymbol{J}_2 \cdot \hat{\boldsymbol{n}}\phi}{\hbar}\right).$$

Base vectors of the whole system
We seek in the product space $\{|j_1m_1\rangle \otimes |j_2m_2\rangle\}$ for the maximal set of commuting operators.
(i) $\boldsymbol{J}_1^2$, $\boldsymbol{J}_2^2$, J_{1z} and J_{2z}.
Their common eigenstates are simply direct products

$$|j_1j_2; m_1m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$

If j_1 and j_2 can be deduced from the context we often denote

$$|m_1m_2\rangle = |j_1j_2; m_1m_2\rangle.$$

The quantum numbers are obtained from the (eigen)equations

$$\begin{aligned}
\boldsymbol{J}_1^2|j_1j_2; m_1m_2\rangle &= j_1(j_1+1)\hbar^2|j_1j_2; m_1m_2\rangle \\
J_{1z}|j_1j_2; m_1m_2\rangle &= m_1\hbar|j_1j_2; m_1m_2\rangle \\
\boldsymbol{J}_2^2|j_1j_2; m_1m_2\rangle &= j_2(j_2+1)\hbar^2|j_1j_2; m_1m_2\rangle \\
J_{2z}|j_1j_2; m_1m_2\rangle &= m_2\hbar|j_1j_2; m_1m_2\rangle.
\end{aligned}$$

(ii) $\boldsymbol{J}^2$, $\boldsymbol{J}_1^2$, $\boldsymbol{J}_2^2$ and J_z.
Their common eigenstate is denoted as

$$|j_1j_2; jm\rangle$$

or shortly

$$|jm\rangle = |j_1j_2;jm\rangle$$

if the quantum numbers j_1 and j_2 can be deduced from the context. The quantum numbers are obtained from the equations

$$\begin{aligned}
\boldsymbol{J}_1^2|j_1j_2;jm\rangle &= j_1(j_1+1)\hbar^2|j_1j_2;jm\rangle \\
\boldsymbol{J}_2^2|j_1j_2;jm\rangle &= j_2(j_2+1)\hbar^2|j_1j_2;jm\rangle \\
\boldsymbol{J}^2|j_1j_2;jm\rangle &= j(j+1)\hbar^2|j_1j_2;jm\rangle \\
J_z|j_1j_2;jm\rangle &= m\hbar|j_1j_2;jm\rangle.
\end{aligned}$$

Now

$$[\boldsymbol{J}^2, J_{1z}] \neq 0, \quad [\boldsymbol{J}^2, J_{2z}] \neq 0,$$

so we cannot add to the set (i) the operator $\boldsymbol{J}^2$, nor to the set (ii) the operators J_{1z} or J_{2z}. Both sets are thus maximal and the corresponding bases complete (and orthonormal), i.e.

$$\begin{aligned}
\sum_{j_1j_2}\sum_{m_1m_2} |j_1j_2;m_1m_2\rangle\langle j_1j_2;m_1m_2| &= 1 \\
\sum_{j_1j_2}\sum_{jm} |j_1j_2;jm\rangle\langle j_1j_2;jm| &= 1.
\end{aligned}$$

In the subspace where the quantum numbers j_1 and j_2 are fixed we have the completeness relations

$$\begin{aligned}
\sum_{m_1m_2} |j_1j_2;m_1m_2\rangle\langle j_1j_2;m_1m_2| &= 1 \\
\sum_{jm} |j_1j_2;jm\rangle\langle j_1j_2;jm| &= 1.
\end{aligned}$$

One can go from the basis (i) to the basis (ii) via the unitary transformation

$$|j_1j_2;jm\rangle = \sum_{m_1m_2} |j_1j_2;m_1m_2\rangle\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle,$$

so also the transformation matrix

$$(C)_{jm,m_1m_2} = \langle j_1j_2;m_1m_2|j_1j_2;jm\rangle$$

is unitary. The elements $\langle j_1j_2;m_1m_2|j_1j_2;jm\rangle$ of the transformation matrix are called *Clebsch-Gordan's coefficients.*
Since

$$J_z = J_{1z} + J_{2z},$$

we must have

$$m = m_1 + m_2,$$

so the Clebsch-Gordan coefficients satisfy the condition

$$\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle = 0, \text{ if } m \neq m_1 + m_2.$$

Further, we must have (QM II)

$$|j_1 - j_2| \leq j \leq j_1 + j_2.$$

It turns out, that the C-G coefficients can be chosen to be real, so the transformation matrix C is in fact orthogonal:

$$\sum_{jm} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \langle j_1 j_2; m_1' m_2' | j_1 j_2; jm\rangle$$
$$= \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$
$$\sum_{m_1 m_2} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \langle j_1 j_2; m_1 m_2 | j_1 j_2; j' m'\rangle$$
$$= \delta_{jj'} \delta_{mm'}.$$

As a special case ($j' = j$ and $m' = m = m_1 + m_2$) we get the normalization condition

$$\sum_{m_1 m_2} |\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle|^2 = 1.$$

Recursion formulas
Operating with the ladder operators to the state $|j_1 j_2; jm\rangle$ we get

$$\begin{aligned} J_\pm |j_1 j_2; jm\rangle = \\ (J_{1\pm} + J_{2\pm}) \sum_{m_1 m_2} |j_1 j_2; m_1 m_2\rangle \\ \times \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle, \end{aligned}$$

or

$$\begin{aligned} &\sqrt{(j \mp m)(j \pm m + 1)} |j_1 j_2; j, m \pm 1\rangle \\ &\quad = \sum_{m_1'} \sum_{m_2'} \Big(\sqrt{(j_1 \mp m_1')(j_1 \pm m_1' + 1)} \\ &\qquad \times |j_1 j_2; m_1' \pm 1, m_2'\rangle \\ &\qquad + \sqrt{(j_2 \pm m_2')(j_2 \pm m_2' + 1)} \\ &\qquad \times |j_1 j_2; m_1', m_2' \pm 1\rangle \Big) \\ &\qquad \times \langle j_1 j_2; m_1' m_2' | j_1 j_2; jm\rangle. \end{aligned}$$

Taking the scalar product on the both sides with the vector $\langle j_1 j_2; m_1 m_2|$ we get

$$\sqrt{(j \mp m)(j \pm m + 1)} \langle j_1 j_2; m_1 m_2 | j_1 j_2; j, m \pm 1\rangle$$

$$
\begin{aligned}
&= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)} \\
&\quad \times \langle j_1 j_2; m_1 \mp 1, m_2 | j_1 j_2; jm\rangle \\
&\quad + \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)} \\
&\quad \times \langle j_1 j_2; m_1, m_2 \mp 1 | j_1 j_2; jm\rangle.
\end{aligned}
$$

The Clebsch-Gordan coefficients are determined uniquely by

1. the recursion formulas.
2. the normalization condition

$$\sum_{m_1 m_2} |\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle|^2 = 1.$$

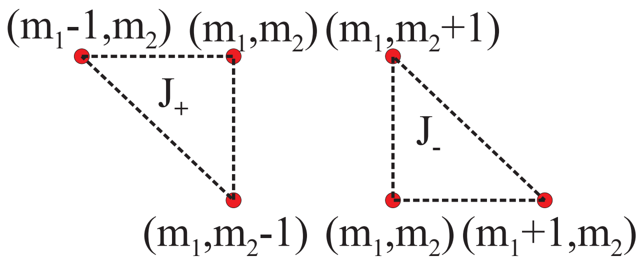
3. the sign conventions, for example

$$\langle j_1 j_2; j'm' | J_{1z} | j_1 j_2; jm\rangle \geq 0.$$

Note Due to the sign conventions the order of the coupling is essential:

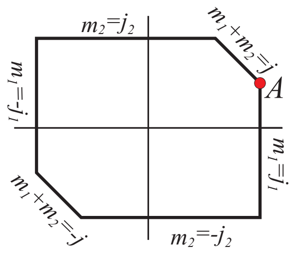
$$|j_1 j_2; jm\rangle = \pm |j_2 j_1; jm\rangle.$$

Graphical representation of recursion formulas

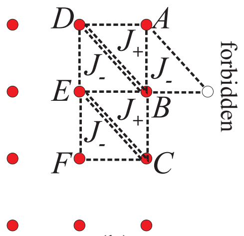


Recursion formula in $m_1 m_2$-plane
We fix j_1, j_2 and j. Then

$$|m_1| \leq j_1, \quad |m_2| \leq j_2, \quad |m_1 + m_2| \leq j.$$



(a)



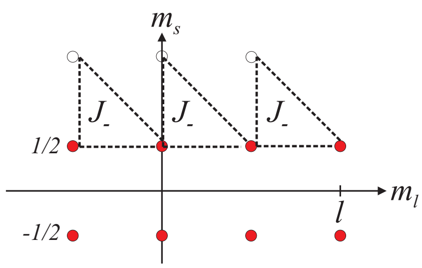
(b)

Using recursion formulas
We see that

1. every C-G coefficient depends on A,
2. the normalization condition determines the absolute value of A,
3. the sign is obtained from the sign conventions.

Example $\boldsymbol{L}+\boldsymbol{S}$-coupling.
Now

$$
\begin{aligned}
j_1 &= l = 0, 1, 2, \ldots \\
m_1 &= m_l = -l, -l+1, \ldots, l-1, l \\
j_2 &= s = \frac{1}{2} \\
m_2 &= m_s = \pm\frac{1}{2} \\
j &= \begin{cases} l \pm \frac{1}{2}, & \text{when } l > 0 \\ \frac{1}{2}, & \text{when } l = 0. \end{cases}
\end{aligned}
$$



Recursion when $j_1 = l$ and $j_2 = s = 1/2$
Using the selection rule

$$m_1 = m_l = m - \frac{1}{2}, \quad m_2 = m_s = \frac{1}{2}$$

and the shorthand notation the J_--recursion gives

$$\begin{aligned}
&\sqrt{(l+\tfrac{1}{2}+m+1)(l+\tfrac{1}{2}-m)}\langle m-\tfrac{1}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m\rangle \\
&\quad = \sqrt{(l+m+\tfrac{1}{2})(l-m+\tfrac{1}{2})} \\
&\quad\quad \times\langle m+\tfrac{1}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m+1\rangle,
\end{aligned}$$

or

$$\langle m-\tfrac{1}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m\rangle = \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}}\langle m+\tfrac{1}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m+1\rangle.$$

Applying the same recursion repeatedly we have

$$\begin{aligned}
&\langle m-\tfrac{1}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m\rangle \\
&\quad = \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}}\sqrt{\frac{l+m+\frac{3}{2}}{l+m+\frac{5}{2}}}\langle m+\tfrac{3}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m+2\rangle \\
&\quad = \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}}\sqrt{\frac{l+m+\frac{3}{2}}{l+m+\frac{5}{2}}}\sqrt{\frac{l+m+\frac{5}{2}}{l+m+\frac{7}{2}}} \\
&\quad\ \langle m+\tfrac{5}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m+3\rangle \\
&\quad = \\
&\quad\ \vdots \\
&\quad = \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}\langle l,\tfrac{1}{2}|l+\tfrac{1}{2},l+\tfrac{1}{2}\rangle.
\end{aligned}$$

If $j = j_{\max} = j_1 + j_2$ and $m = m_{\max} = j_1 + j_2$ one must have

$$\begin{aligned}
&|j_1j_2;jm\rangle = \\
&\quad \langle j_1j_2;m_1=j_1,m_2=j_2|j_1j_2;jm\rangle|j_1m_1\rangle|j_2m_2\rangle.
\end{aligned}$$

Now the normalization condition

$$|\langle j_1j_2;m_1=j_1,m_2=j_2|j_1j_2;jm\rangle|^2 = 1$$

and the sign convention give

$$\langle j_1j_2;m_1=j_1,m_2=j_2|j_1j_2;jm\rangle = 1.$$

Thus, in the case of the spin-orbit coupling,

$$\langle l,\tfrac{1}{2}|l+\tfrac{1}{2},l+\tfrac{1}{2}\rangle = 1,$$

or

$$\langle m-\tfrac{1}{2},\tfrac{1}{2}|l+\tfrac{1}{2},m\rangle = \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}.$$

With the help of the recursion relations, normalization condition and sign convention the rest of the C-G coefficients can be evaluated, too. We get

$$\begin{pmatrix} |j=l+\frac{1}{2},m\rangle \\ |j=l-\frac{1}{2},m\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}} & \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}} \\ -\sqrt{\frac{l-m+\frac{1}{2}}{2l+1}} & \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}} \end{pmatrix}$$
$$\begin{pmatrix} |m_l=m-\frac{1}{2},m_s=\frac{1}{2}\rangle \\ |m_l=m+\frac{1}{2},m_s=-\frac{1}{2}\rangle \end{pmatrix}.$$

Rotation matrices
If $\mathcal{D}^{(j_1)}(R)$ is a rotation matrix in the base $\{|j_1m_1\rangle|m_1=-j_1,\ldots,j_1\}$ and $\mathcal{D}^{(j_2)}(R)$ a rotation matrix in the base $\{|j_2m_2\rangle|m_2=-j_2,\ldots,j_2\}$, then $\mathcal{D}^{(j_1)}(R)\otimes\mathcal{D}^{(j_2)}(R)$ is a rotation matrix in the $(2j_1+1)\times(2j_2+1)$-dimensional base $\{|j_1,m_1\rangle\otimes|j_2,m_2\rangle\}$. Selecting suitable superpositions of the vectors $|j_1,m_1\rangle\otimes|j_2,m_2\rangle$ the matrix takes the form like

$$\mathcal{D}^{(j_1)}(R)\otimes\mathcal{D}^{(j_2)}(R)\longrightarrow$$
$$\begin{pmatrix} \boxed{\mathcal{D}^{(j_1+j_2)}} & & & 0 \\ & \boxed{\mathcal{D}^{(j_1+j_2-1)}} & & \\ & & \ddots & \\ 0 & & & \boxed{\mathcal{D}^{(|j_1-j_2|)}} \end{pmatrix}.$$

One can thus write

$$\mathcal{D}^{(j_1)}\otimes\mathcal{D}^{(j_2)} = \mathcal{D}^{(j_1+j_2)}\oplus\mathcal{D}^{(j_1+j_2-1)}\oplus\cdots\oplus\mathcal{D}^{(|j_1-j_2|)}.$$

As a check we can calculate the dimensions:

$$\begin{aligned}(2j_1+1)(2j_2+1) &= \\ &2(j_1+j_2)+1+2(j_1+j_2-1)+1 \\ &+\cdots+2|j_1-j_2|+1.\end{aligned}$$

The matrix elements of the rotation operator satisfy

$$\begin{aligned}&\langle j_1j_2;m_1m_2|\mathcal{D}(R)|j_1j_2;m_1'm_2'\rangle \\ &\quad=\langle j_1m_1|\mathcal{D}(R)|j_1m_1'\rangle\langle j_2m_2|\mathcal{D}(R)|j_2m_2'\rangle \\ &\quad=\mathcal{D}^{(j_1)}_{m_1m_1'}(R)\mathcal{D}^{(j_2)}_{m_2m_2'}(R).\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\langle j_1 j_2; m_1 m_2 | \mathcal{D}(R) | j_1 j_2; m_1' m_2' \rangle \\
&\quad = \sum_{jm} \sum_{j'm'} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm \rangle \\
&\quad\quad \times \langle j_1 j_2; jm | \mathcal{D}(R) | j_1 j_2; j'm' \rangle \\
&\quad\quad \times \langle j_1 j_2; j'm' | j_1 j_2; m_1' m_2' \rangle \\
&\quad = \sum_{jm} \sum_{j'm'} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm \rangle \mathcal{D}^{(j)}_{mm'}(R) \delta_{jj'} \\
&\quad\quad \times \langle j_1 j_2; m_1' m_2' | j_1 j_2; j'm' \rangle .
\end{aligned}$$

We end up with the *Clebsch-Gordan series*

$$\begin{aligned}
&\mathcal{D}^{(j_1)}_{m_1 m_1'}(R) \mathcal{D}^{(j_2)}_{m_2 m_2'}(R) = \\
&\quad \sum_{j} \sum_{m} \sum_{m'} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm \rangle \\
&\quad \times \langle j_1 j_2; m_1' m_2' | j_1 j_2; jm' \rangle \mathcal{D}^{(j)}_{mm'}(R).
\end{aligned}$$

As an application we have

$$\begin{aligned}
&\int d\Omega Y_l^{m*}(\theta,\phi) Y_{l_1}^{m_1}(\theta,\phi) Y_{l_2}^{m_2}(\theta,\phi) \\
&\quad = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} \\
&\quad\quad \times \langle l_1 l_2; 00 | l_1 l_2; l0 \rangle \langle l_1 l_2; m_1 m_2 | l_1 l_2; lm \rangle .
\end{aligned}$$