$3j$- $6j$- and $9j$-symbols

It is obvious by now that coupling of many angular momenta ($n >> 2$) can be difficult and time-consuming.
We are searching for a compact notation for the angular momentum addition rules in different cases

i) coupling of 2 angular momenta $J = j_1 + j_2$
(3j symbol)

ii) coupling of 3 angular momenta $J = j_{12} + j_3 = (j_1 + j_2) + j_3$
(6j symbol)

ii) coupling of 4 angular momenta $J = j_{123} + j_4$
(9j symbol)

The Clebsch-Gordan coefficients obey certain symmetry relations, like

$$\begin{aligned}
&\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle \\
&\qquad = (-1)^{j_1+j_2-j} \langle j_2 j_1; m_2 m_1 | j_2 j_1; jm\rangle \\
&\langle j_1 j_2; m_1 m_2 | j_1 j_2; j_3 m_3\rangle \\
&\qquad = (-1)^{j_2+m_2} \sqrt{\frac{2j_3+1}{2j_1+1}} \langle j_2 j_3; -m_2, m_3 | j_2 j_3; j_1 m_1\rangle \\
&\langle j_1 j_2; m_1 m_2 | j_1 j_2; j_3 m_3\rangle \\
&\qquad = (-1)^{j_1-m_1} \sqrt{\frac{2j_3+1}{2j_2+1}} \langle j_3 j_1; m_3, -m_1 | j_3 j_1; j_2 m_2\rangle \\
&\langle j_1 j_2; m_1 m_2 | j_1 j_2; j_3 m_3\rangle \\
&\qquad = (-1)^{j_1+j_2-j_3} \langle j_1 j_2; -m_1, -m_2 | j_1 j_2; j_3, -m_3\rangle .
\end{aligned}$$

Note The first relation shows that the coupling order is essential.
We define more symmetric $3j$-*symbols*:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \frac{(-1)^{j_1-j_2-m_2}}{\sqrt{2j_3+1}} \langle j_1 j_2; m_1 m_2 | j_1 j_2; j_3, -m_3\rangle .$$

They satisfy

$$\begin{aligned}
&\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\
&\qquad = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} \\
&\qquad = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}
\end{aligned}$$

$$= (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}$$
$$= (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}$$
$$= (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}$$

For example, we see that the coefficients

$$\begin{pmatrix} \frac{3}{2} & \frac{3}{2} & 2 \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 3 \\ 1 & 1 & -2 \end{pmatrix}.$$

vanish.
On the other hand, the orthogonality properties are somewhat more complicated:

$$\sum_{j_3} \sum_{m_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix}$$
$$= \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

and

$$\sum_{m_1} \sum_{m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix}$$
$$= \frac{\delta_{j_3 j_3'} \delta_{m_3 m_3'} \delta(j_1 j_2 j_3)}{\sqrt{2j_3+1}},$$

where

$$\delta(j_1 j_2 j_3) = \begin{cases} 1, & \text{when } |j_1 - j_2| \le j_3 \le j_1 + j_2 \\ 0, & \text{otherwise.} \end{cases}$$

$6j$-symbols
Let us couple three angular momenta, j_1, j_2 and j_3, to the angular momentum J. There are two ways:

1. first $j_1, j_2 \longrightarrow j_{12}$ and then $j_{12}, j_3 \longrightarrow J$.
2. first $j_2, j_3 \longrightarrow j_{23}$ and then $j_{23}, j_1 \longrightarrow J$.

Let's choose the first way. The quantum number j_{12} must satisfy the selection rules

$$\begin{aligned} |j_1 - j_2| &\le j_{12} \le j_1 + j_2 \\ |j_{12} - j_3| &\le J \le j_{12} + j_3. \end{aligned}$$

The states belonging to different j_{12} are independent so we must specify the intermediate state j_{12}. We use the notation

$$|(j_1 j_2) j_{12} j_3; JM\rangle.$$

Explicitely one has

$$
\begin{aligned}
&|(j_1j_2)j_{12}j_3;JM\rangle \\
&\quad = \sum_{m_{12}}\sum_{m_3}|j_1j_2;j_{12}m_{12}\rangle|j_3m_3\rangle \\
&\quad\quad \times\langle j_{12}j_3;m_{12}m_3|j_{12}j_3;JM\rangle \\
&\quad = \sum_{m_1m_2m_3m_{12}}|j_1m_1\rangle|j_2m_2\rangle|j_3m_3\rangle \\
&\quad\quad \times\langle j_1j_2;m_1m_2|j_1j_2;j_{12}m_{12}\rangle \\
&\quad\quad \times\langle j_{12}j_3;m_{12}m_3|j_{12}j_3;JM\rangle.
\end{aligned}
$$

Correspondingly the angular momenta coupled in way 2 satisfy

$$
\begin{aligned}
&|j_1(j_2j_3)j_{23};JM\rangle \\
&\quad = \sum_{m_{23}}\sum_{m_1}|j_1m_1\rangle|j_2j_3;j_{23}m_{23}\rangle \\
&\quad\quad \times\langle j_1j_{23};m_1m_{23}|j_1j_{23};JM\rangle \\
&\quad = \sum_{m_1m_2m_3m_{23}}|j_1m_1\rangle|j_2m_2\rangle|j_3m_3\rangle \\
&\quad\quad \times\langle j_2j_3;m_2m_3|j_2j_3;j_{23}m_{23}\rangle \\
&\quad\quad \times\langle j_1j_{23};m_1m_{23}|j_1j_{23};JM\rangle.
\end{aligned}
$$

Both bases are complete so there is a unitary transform between them:

$$
\begin{aligned}
|j_1(j_2j_3)j_{23};JM\rangle &= \sum_{j_{12}}|(j_1j_2)j_{12}j_3;JM\rangle \\
&\times\langle (j_1j_2)j_{12}j_3;JM|j_1(j_2j_3)j_{23};JM\rangle.
\end{aligned}
$$

these are the *recoupling coefficients* for whom it is not necessary to show the quantum number M, because

Theorem 1 *In the transformation*

$$|\alpha;jm\rangle = \sum_{\beta}|\beta;jm\rangle\langle\beta;jm|\alpha;jm\rangle$$

the coefficients $\langle\beta;jm|\alpha;jm\rangle$ *do not depend on the quantum number* m.

Proof: Let us suppose that $m<j$. Now

$$|\alpha;j,m+1\rangle = \sum_{\beta}|\beta;j,m+1\rangle\langle\beta;j,m+1|\alpha;j,m+1\rangle.$$

On the other hand

$$
\begin{aligned}
|\alpha;j,m+1\rangle &= \frac{J_+}{\hbar\sqrt{(j+m+1)(j-m)}}|\alpha;jm\rangle \\
&= \sum_{\beta}|\beta;j,m+1\rangle\langle\beta;jm|\alpha;jm\rangle,
\end{aligned}
$$

so

$$\langle \beta; j, m+1 | \alpha; j, m+1 \rangle = \langle \beta; j, m | \alpha; j, m \rangle \quad \blacksquare$$

The explicit expression for the recoupling coefficients will be

$$\begin{aligned} &\langle (j_1 j_2) j_{12} j_3; J | j_1 (j_2 j_3) j_{23}; J \rangle \\ &\quad = \sum_{\substack{m_1 m_2 m_3 \\ m_{12} m_{23}}} \langle j_{12} j_3; JM | j_{12} j_3; m_{12} m_3 \rangle \\ &\qquad \times \langle j_1 j_2; j_{12} m_{12} | j_1 j_2; m_1 m_2 \rangle \\ &\qquad \times \langle j_2 j_3; m_2 m_3 | j_2 j_3; j_{23} m_{23} \rangle \\ &\qquad \times \langle j_1 j_{23}; m_1 m_{23} | j_1 j_{23}; JM \rangle. \end{aligned}$$

We define the more symmetric $6j$*-symbols*:

$$\begin{aligned} &\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix} \\ &\quad \equiv \frac{(-1)^{j_1+j_2+j_3+J}}{\sqrt{(2j_{12}+1)(2j_{23}+1)}} \\ &\qquad \times \langle (j_1 j_2) j_{12} j_3; J | j_1 (j_2 j_3) j_{23}; J \rangle \\ &\quad = \frac{(-1)^{j_1+j_2+j_3+J}}{\sqrt{(2j_{12}+1)(2j_{23}+1)}} \\ &\qquad \times \sum_{m_1 m_2} \langle j_1 j_2; m_1 m_2 | j_1 j_2; j_{12}, m_1+m_2 \rangle \\ &\qquad \times \langle j_{12} j_3; m_1+m_2, M-m_1-m_2 | j_{12} j_3; JM \rangle \\ &\qquad \times \langle j_2 j_3; m_2, M-m_1-m_2 | j_2 j_3; j_{23}, M-m_1 \rangle \\ &\qquad \times \langle j_1 j_{23}; m_1, M-m_1 | j_1 j_{23}; JM \rangle. \end{aligned}$$

The *triads* (any 3 j combination going from left to right) satisfy the triangular condition.
Permutation of columns leaves the symbol invariant.
2 corresponding elements on rows can be exchanged leaving the symbol invariant.
We can handle analogously the coupling of 4 angular momenta.
Transformations from a coupling scheme to another are mediated by the $9j$*-symbols*:

$$\begin{aligned} &\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix} \\ &\quad \equiv \frac{\langle (j_1 j_2) j_{12} (j_3 j_4) j_{34}; j | (j_1 j_3) j_{13} (j_2 j_4) j_{24}; j \rangle}{\sqrt{(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1}}. \end{aligned}$$