Tensor operators

We have used the vector notation for three component operators for example to express the scalar product, like

$$\boldsymbol{p}\cdot\boldsymbol{x}' = p_x x' + p_y y' + p_z z'.$$

Classically a vector is a quantity that under rotations transforms like $\boldsymbol{V}\in\mathcal{R}^3$ (or $\in\mathcal{C}^3$), i.e. if $R\in\mathrm{O}(3)$, then

$$V_i' = \sum_{j=1}^{3} R_{ij}V_j.$$

In quantum mechanics $\boldsymbol{V}$ is a *vector operator* provided that $\langle\boldsymbol{V}\rangle\in\mathcal{C}^3$ is a vector:

$$\begin{aligned} {}_R\langle\alpha|V_i|\alpha\rangle_R &= \langle\alpha|\mathcal{D}^\dagger(R)V_i\mathcal{D}(R)|\alpha\rangle \\ &= \sum_{j=1}^{3} R_{ij}\langle\alpha|V_j|\alpha\rangle, \\ &\qquad \forall|\alpha\rangle\in\mathcal{H}, R\in\mathrm{O}(3). \end{aligned}$$

Thus we must have

$$\mathcal{D}^\dagger(R)V_i\mathcal{D}(R) = \sum_j R_{ij}V_j.$$

Thus the infinitesimal rotations

$$\mathcal{D}(\hat{\boldsymbol{n}}\epsilon) = 1 - \frac{i\epsilon\boldsymbol{J}\cdot\hat{\boldsymbol{n}}}{\hbar}$$

satisfy

$$\begin{aligned} &\left(1+\frac{i\epsilon\boldsymbol{J}\cdot\hat{\boldsymbol{n}}}{\hbar}\right)V_i\left(1-\frac{i\epsilon\boldsymbol{J}\cdot\hat{\boldsymbol{n}}}{\hbar}\right) \\ &\quad = V_i + \frac{i\epsilon}{\hbar}\left(\boldsymbol{J}\cdot\hat{\boldsymbol{n}}V_i - V_i\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\right) + \mathcal{O}(\epsilon^2) \\ &\quad = \sum_j R_{ij}V_j \end{aligned}$$

or

$$V_i + \frac{\epsilon}{i\hbar}[V_i, \boldsymbol{J}\cdot\hat{\boldsymbol{n}}] = \sum_j R_{ij}(\hat{\boldsymbol{n}}\epsilon)V_j.$$

Substituting the explicit expressions for infinitesimal rotations, like

$$R(\hat{\boldsymbol{z}}\epsilon) = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1-\frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we get

$$V_x + \frac{\epsilon}{i\hbar}[V_x, J_z] = V_x - \epsilon V_y + \mathcal{O}(\epsilon^3).$$

Handling similarly the other components we end up with

$$\boxed{[V_i, J_j] = i\hbar\epsilon_{ijk}V_k.}$$

Finite rotation
A finite rotation specified by Euler angles is accomplished by rotating around coordinate axises, so we have to consider such expressions as

$$\exp\left(\frac{iJ_j\phi}{\hbar}\right)V_i\exp\left(-\frac{iJ_j\phi}{\hbar}\right).$$

Applying the Baker-Hausdorff lemma

$$\begin{aligned} &e^{iG\lambda}Ae^{-iG\lambda} = \\ &\quad A + i\lambda[G, A] + \left(\frac{i^2\lambda^2}{2!}\right)[G, [G, A]] + \cdots \\ &\quad + \left(\frac{i^n\lambda^n}{n!}\right)[G, [G, [G, \ldots [G, A]]]\ldots] + \cdots \end{aligned}$$

we end up with the commutators

$$[J_j, [J_j, [\cdots[J_j, V_i]\cdots]]].$$

These will be evaluated in turn into operators V_i and V_k $(k \neq i, j)$.
A vector operator $(\boldsymbol{V})$ is *defined* so that it satisfies the commutation relation

$$[V_i, J_j] = i\hbar\epsilon_{ijk}V_k.$$

We can easily see that for example $\boldsymbol{p}$, $\boldsymbol{x}$ and $\boldsymbol{J}$ are vector operators.

In classical mechanics a quantity which under rotations transforms like

$$T_{\underbrace{ijk\cdots}_{n \text{ indeces}}} \longrightarrow \sum_{i'}\sum_{j'}\sum_{k'}\cdots R_{ii'}R_{jj'}R_{kk'}\cdots T_{i'j'k'\ldots},$$

is called a Cartesian tensor of the rank n.
Example The dyad product of the vectors $\boldsymbol{U}$ and $\boldsymbol{V}$: $(U_1, U_2, U_3)^T \times (V_1, V_2, V_3)$

$$T_{ij} = U_iV_j$$

is a tensor of rank 2.

Cartesian tensors are reducible, for example the dyad product can be written as

$$\begin{aligned} U_i V_j &= \frac{\boldsymbol{U}\cdot\boldsymbol{V}}{3}\delta_{ij} + \frac{(U_iV_j - U_jV_i)}{2} \\ &\quad + \left(\frac{U_iV_j + U_jV_i}{2} - \frac{\boldsymbol{U}\cdot\boldsymbol{V}}{3}\delta_{ij}\right). \end{aligned}$$

We see that the terms transform under rotations differently:

- $\frac{\boldsymbol{U}\cdot\boldsymbol{V}}{3}\delta_{ij}$ is invariant scalar. There is 1 term.
- $\frac{(U_iV_j - U_jV_i)}{2}$ retains its antisymmetry (anti-symmetric tensor). There are 3 terms.
- $\left(\frac{U_iV_j + U_jV_i}{2} - \frac{\boldsymbol{U}\cdot\boldsymbol{V}}{3}\delta_{ij}\right)$ retains its symmetry and tracelessness. There are 5 terms.

We recognize that the number of terms checks (1+3+5=9) and that the partition might have something to do with the angular momentum since the multiplicities correspond to the multiplicities of the angular momenta $l = 0, 1, 2$.
We define the *spherical tensor* $T_q^{(k)}$ of rank k so that the argument $\hat{\boldsymbol{n}}$ of the spherical function

$$Y_l^m(\hat{\boldsymbol{n}}) = \langle \hat{\boldsymbol{n}} | lm \rangle$$

is replaced by the vector $\boldsymbol{V}$:

$$T_q^{(k)} = Y_{l=k}^{m=q}(\boldsymbol{V}).$$

Example The spherical function Y_1:

$$\begin{aligned} Y_1^0 &= \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \mapsto T_0^{(1)} = \sqrt{\frac{3}{4\pi}}V_z \\ Y_1^{\pm 1} &= \mp\sqrt{\frac{3}{4\pi}}\frac{x \pm iy}{\sqrt{2}r} \mapsto T_{\pm 1}^{(1)} = \sqrt{\frac{3}{4\pi}}\left(\mp\frac{V_x \pm iV_y}{\sqrt{2}}\right). \end{aligned}$$

Similarly we could construct for example a spherical tensor of rank 2:

$$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\frac{(x \pm iy)^2}{r^2} \mapsto T_{\pm 2}^{(2)} = \sqrt{\frac{15}{32\pi}}(V_x \pm iV_y)^2.$$

The tensors $T_q^{(k)}$ are irreducible, i.e. there does not exist any proper subset

$$\{T_{p_1}^{(k)}, T_{p_2}^{(k)}, \ldots\} \subset \{T_q^{(k)} | q = -k, \ldots, +k\},$$

which would remain invariant under rotations.

Transformation of spherical tensors

Under the rotation R an eigenstate of the direction transforms like

$$|\hat{\boldsymbol{n}}\rangle \longrightarrow |\hat{\boldsymbol{n}}'\rangle = \mathcal{D}(R)|\hat{\boldsymbol{n}}\rangle.$$

The state vectors $|lm\rangle$, on the other hand, transform under the rotation R^{-1} like

$$\mathcal{D}(R^{-1})|l,m\rangle = \sum_{m'} |l,m'\rangle \mathcal{D}^{(l)}_{m'm}(R^{-1}).$$

So we get

$$\begin{aligned} Y_l^m(\hat{\boldsymbol{n}}') &= \langle \hat{\boldsymbol{n}}'|lm\rangle = \langle \hat{\boldsymbol{n}}|\mathcal{D}^\dagger(R)|lm\rangle \\ &= \langle \hat{\boldsymbol{n}}|\mathcal{D}(R^{-1})|lm\rangle = \sum_{m'} \langle \hat{\boldsymbol{n}}|lm'\rangle \mathcal{D}^{(l)}_{m'm}(R^{-1}) \\ &= \sum_{m'} Y_l^{m'}(\hat{\boldsymbol{n}}) \mathcal{D}^{(l)}_{m'm}(R^{-1}) \\ &= \sum_{m'} Y_l^{m'}(\hat{\boldsymbol{n}}) \mathcal{D}^{(l)}_{mm'}{}^*(R). \end{aligned}$$

We define a tensor operator $Y_l^m(\boldsymbol{V})$ so that

$$\mathcal{D}^\dagger(R) Y_l^m(\boldsymbol{V}) \mathcal{D}(R) = \sum_{m'} Y_l^{m'}(\boldsymbol{V}) \mathcal{D}^{(l)^*}_{mm'}(R).$$

Generalizing we define: $T_q^{(k)}$ is a $(2k+1)$-component spherical tensor of rank k if and only if

$$\mathcal{D}^\dagger(R) T_q^{(k)} \mathcal{D}(R) = \sum_{q'=-k}^{k} \mathcal{D}^{(k)^*}_{qq'}(R) T_{q'}^{(k)}$$

or equivalently

$$\boxed{\mathcal{D}(R) T_q^{(k)} \mathcal{D}^\dagger(R) = \textstyle\sum_{q'=-k}^{k} \mathcal{D}^{(k)}_{q'q}(R) T_{q'}^{(k)}.}$$

Under the infinitesimal rotations

$$\mathcal{D}(\hat{\boldsymbol{n}}\epsilon) = \left(1 - \frac{i\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\epsilon}{\hbar}\right)$$

a spherical tensor behaves thus like

$$\begin{aligned} &\left(1 + \frac{i\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\epsilon}{\hbar}\right) T_q^{(k)} \left(1 - \frac{i\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\epsilon}{\hbar}\right) \\ &\quad = \sum_{q'=-k}^{k} T_{q'}^{(k)} \langle kq'| \left(1 + \frac{i\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\epsilon}{\hbar}\right) |kq\rangle \\ &\quad = \sum_{q'=-k}^{k} T_{q'}^{(k)} \langle kq'|kq\rangle + \sum_{q'=-k}^{k} i\epsilon T_{q'}^{(k)} \langle kq'|\boldsymbol{J}\cdot\hat{\boldsymbol{n}}|kq\rangle, \end{aligned}$$

or

$$[\boldsymbol{J}\cdot\hat{\boldsymbol{n}}, T_q^{(k)}] = \sum_{q'} T_{q'}^{(k)} \langle kq'|\boldsymbol{J}\cdot\hat{\boldsymbol{n}}|kq\rangle.$$

Choosing $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}}$ we get

$$\boxed{[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}}$$

and

$$\boxed{[J_\pm, T_q^{(k)}] = \hbar\sqrt{(k\mp q)(k\pm q+1)}T_{q\pm1}^{(k)}.}$$

Example Decomposition of the dyad product.
We form spherical tensors of rank 1 from the vector operators $\boldsymbol{U}$ and $\boldsymbol{V}$:

$$\begin{aligned} U_0 &= U_z, & V_0 &= V_z, \\ U_{\pm1} &= \mp\frac{U_x \pm iU_y}{\sqrt{2}}, & V_{\pm1} &= \mp\frac{V_x \pm iV_y}{\sqrt{2}}. \end{aligned}$$

Now

$$\begin{aligned} T_0^{(0)} &= -\frac{\boldsymbol{U}\cdot\boldsymbol{V}}{3} = \frac{U_{+1}V_{-1} + U_{-1}V_{+1} - U_0V_0}{3}, \\ T_q^{(1)} &= \frac{(\boldsymbol{U}\times\boldsymbol{V})_q}{i\sqrt{2}}, \\ T_{\pm2}^{(2)} &= U_{\pm1}V_{\pm1}, \\ T_{\pm1}^{(2)} &= \frac{U_{\pm1}V_0 + U_0V_{\pm1}}{\sqrt{2}}, \\ T_0^{(2)} &= \frac{U_{+1}V_{-1} + 2U_0V_0 + U_{-1}V_{+1}}{\sqrt{6}}. \end{aligned}$$

In general we have
Theorem 1 *Let $X_{q_1}^{(k_1)}$ and $Z_{q_2}^{(k_2)}$ be irreducible spherical tensors of rank k_1 and k_2. Then*

$$T_q^{(k)} = \sum_{q_1}\sum_{q_2} \langle k_1k_2; q_1q_2|k_1k_2; kq\rangle X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)}$$

is a (irreducible) spherical tensor of rank k.
Proof: We show that $T_q^{(k)}$ transforms like

$$\mathcal{D}^\dagger(R) T_q^{(k)} \mathcal{D}(R) = \sum_{q'=-k}^{k} \mathcal{D}_{qq'}^{(k)^*}(R) T_{q'}^{(k)}.$$

Now

$$\mathcal{D}^\dagger(R) T_q^{(k)} \mathcal{D}(R)$$

$$
\begin{aligned}
&= \sum_{q_1} \sum_{q_2} \langle k_1 k_2; q_1 q_2 | k_1 k_2; kq \rangle \\
&\times \mathcal{D}^\dagger(R) X_{q_1}^{(k_1)} \mathcal{D}(R) \mathcal{D}^\dagger(R) Z_{q_2}^{(k_2)} \mathcal{D}(R) \\
&= \sum_{q_1} \sum_{q_2} \sum_{q_1'} \sum_{q_2'} \langle k_1 k_2; q_1 q_2 | k_1 k_2; kq \rangle \\
&\times X_{q_1'}^{(k_1)} \mathcal{D}_{q_1' q_1}^{(k_1)}(R^{-1}) Z_{q_2'}^{(k_2)} \mathcal{D}_{q_2' q_2}^{(k_2)}(R^{-1}) \\
&= \sum_{k''} \sum_{q_1} \sum_{q_2} \sum_{q_1'} \sum_{q_2'} \sum_{q''} \sum_{q'} \langle k_1 k_2; q_1 q_2 | k_1 k_2; kq \rangle \\
&\times \langle k_1 k_2; q_1' q_2' | k_1 k_2; k'' q' \rangle \\
&\times \langle k_1 k_2; q_1 q_2 | k_1 k_2; k'' q'' \rangle \mathcal{D}_{q' q''}^{(k'')}(R^{-1}) X_{q_1'}^{(k_1)} Z_{q_2'}^{(k_2)},
\end{aligned}
$$

where we have substituted the Clebsch-Gordan series expansion

$$
\begin{aligned}
&\mathcal{D}_{m_1 m_1'}^{(j_1)}(R) \mathcal{D}_{m_2 m_2'}^{(j_2)}(R) = \\
&\qquad \sum_{j} \sum_{m} \sum_{m'} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm \rangle \\
&\qquad \times \langle j_1 j_2; m_1' m_2' | j_1 j_2; jm' \rangle \mathcal{D}_{mm'}^{(j)}(R)
\end{aligned}
$$

Taking into account the orthogonality of the Clebsch-Gordan coefficients

$$
\begin{aligned}
\sum_{m_1 m_2} \langle j_1 j_2; m_1 m_2 | j_1 j_2; jm \rangle \langle j_1 j_2; m_1 m_2 | j_1 j_2; j' m' \rangle& \\
= \delta_{jj'} \delta_{mm'}&
\end{aligned}
$$

we get

$$
\begin{aligned}
&\mathcal{D}^\dagger(R) T_q^{(k)} \mathcal{D}(R) \\
&\quad = \sum_{k''} \sum_{q_1'} \sum_{q_2'} \sum_{q''} \sum_{q'} \delta_{kk''} \delta_{qq''} \langle k_1 k_2; q_1' q_2' | k_1 k_2; k'' q' \rangle \\
&\quad \times \mathcal{D}_{q' q''}^{(k'')}(R^{-1}) X_{q_1'}^{(k_1)} Z_{q_2'}^{(k_2)},
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
&\mathcal{D}^\dagger(R) T_q^{(k)} \mathcal{D}(R) \\
&\quad = \sum_{q'} \left(\sum_{q_1' q_2'} \langle k_1 k_2; q_1' q_2' | k_1 k_2; kq' \rangle X_{q_1'}^{(k_1)} Z_{q_2'}^{(k_2)} \right) \\
&\quad \times \mathcal{D}_{q' q}^{(k)}(R^{-1}) \\
&\quad = \sum_{q'} T_{q'}^{(k)} \mathcal{D}_{q' q}^{(k)}(R^{-1}) = \sum_{q'} \mathcal{D}_{qq'}^{(k)^*}(R) T_{q'}^{(k)} \quad \blacksquare
\end{aligned}
$$

Matrix elements of tensor operators

Theorem 2 *The matrix elements of the tensor operator* $T_q^{(k)}$ *satisfy*

$$\langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = 0,$$

unless $m' = q + m$.

Proof: Due to the property

$$[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}$$

we have

$$\begin{aligned}&\langle \alpha', j'm'|[J_z, T_q^{(k)}] - \hbar q T_q^{(k)}|\alpha, jm\rangle \\ &\quad = [(m' - m)\hbar - q\hbar] \times \langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = 0,\end{aligned}$$

so

$$\langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = 0,$$

if $m' \neq q + m$ ■

Theorem 3 *(*Wigner-Eckardt's theorem*) The matrix elements of a tensor operator between eigenstates of the angular momentum satisfy the relation*

$$\langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = \langle jk; mq|jk; j'm'\rangle \frac{\langle \alpha' j' \| T^{(k)} \| \alpha j\rangle}{\sqrt{2j+1}},$$

where the reduced matrix element $\langle \alpha' j' \| T^{(k)} \| \alpha j\rangle$ *depends neither on the quantum numbers* m, m' *nor on* q.

Proof: Since $T_q^{(k)}$ is a tensor operator it satisfies the condition

$$[J_\pm, T_q^{(k)}] = \hbar\sqrt{(k \mp q)(k \pm q + 1)}\, T_{q\pm 1}^{(k)},$$

so

$$\begin{aligned}&\langle \alpha', j'm'|[J_\pm, T_q^{(k)}]|\alpha, jm\rangle \\ &\quad = \hbar\sqrt{(k \mp q)(k \pm q + 1)}\langle \alpha', j'm'|T_{q\pm 1}^{(k)}|\alpha, jm\rangle.\end{aligned}$$

Substituting the matrix elements of the ladder operators we get

$$\begin{aligned}&\sqrt{(j' \pm m')(j' \mp m' + 1)}\langle \alpha', j', m' \mp 1|T_q^{(k)}|\alpha, jm\rangle \\ &\quad = \sqrt{(j \mp m)(j \pm m + 1)}\langle \alpha', j', m'|T_q^{(k)}|\alpha, j, m \pm 1\rangle \\ &\quad + \sqrt{(k \mp q)(k \pm q + 1)}\langle \alpha', j', m'|T_{q\pm 1}^{(k)}|\alpha, jm\rangle.\end{aligned}$$

If we now substituted $j' \to j$, $m' \to m$, $j \to j_1$, $m \to m_1$, $k \to j_2$ and $q \to m_2$, we would note that the recursion formula above is exactly like the recursion formula for the Clebsch-Gordan coefficients,

$$\begin{aligned}
&\sqrt{(j \mp m)(j \pm m + 1)}\langle j_1 j_2; m_1 m_2 | j_1 j_2; j, m \pm 1\rangle \\
&= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)} \\
&\quad \times \langle j_1 j_2; m_1 \mp 1, m_2 | j_1 j_2; jm\rangle \\
&\quad + \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)} \\
&\quad \times \langle j_1 j_2; m_1, m_2 \mp 1 | j_1 j_2; jm\rangle.
\end{aligned}$$

Both recursions are of the form $\sum_j a_{ij} x_j = 0$, or sets of linear homogenous simultaneous equations with the same coefficients a_{ij}. So we have two sets of equations

$$\sum_j a_{ij} x_j = 0, \quad \sum_j a_{ij} y_j = 0,$$

one for the matrix elements (x_i) of the tensor operator and the other for the Clebsch-Gordan coefficients (y_i). These sets of equations tell that

$$\frac{x_j}{x_k} = \frac{y_j}{y_k} \quad \forall j \text{ and } k \text{ fixed},$$

so $x_j = c y_j$ while c is a proportionality coefficient independent of the indeces j. Thus we see that

$$\begin{aligned}
&\langle \alpha', j'm' | T_q^{(k)} | \alpha, jm\rangle \\
&\quad = (\text{ constant independent on } m,\, q \text{ and } m') \\
&\quad\quad \times \langle jk; mq | jk; j'm'\rangle.
\end{aligned}$$

If we write the proportionality coefficient like

$$\frac{\langle \alpha' j' \| T^{(k)} \| \alpha j\rangle}{\sqrt{2j+1}}$$

we are through. ■
According to the Wigner-Eckart theorem a matrix element of a tensor operator is a product of two factors, of which
$\langle jk; mq | jk; j'm'\rangle$ depends only on the geometry, i.e. on the orientation of the system with respect to the z-axis (no reference to the operator).
$\frac{\langle \alpha' j' \| T^{(k)} \| \alpha j\rangle}{\sqrt{2j+1}}$ depends on the dynamics of the system (but not on the magnetic quantum numbers).
Thus, we can evaluate the tensor part with one configuration and the rest follows from geometry.
As a special case we have the *projection theorem*:
Theorem 4 *Let*

$$J_{\pm 1} = \mp \frac{1}{\sqrt{2}}(J_x \pm i J_y) = \mp \frac{1}{\sqrt{2}} J_{\pm}, \quad J_0 = J_z$$

be the components of the tensor operator corresponding to the angular momentum. Then

$$\langle \alpha', jm'|V_q|\alpha, jm\rangle = \frac{\langle \alpha', jm|\boldsymbol{J}\cdot\boldsymbol{V}|\alpha, jm\rangle}{\hbar^2 j(j+1)}\langle jm'|J_q|jm\rangle.$$

Proof: Due to the expansions

$$\begin{aligned}
T_0^{(0)} &= -\frac{\boldsymbol{U}\cdot\boldsymbol{V}}{3} = \frac{U_{+1}V_{-1} + U_{-1}V_{+1} - U_0V_0}{3}, \\
T_q^{(1)} &= \frac{(\boldsymbol{U}\times\boldsymbol{V})_q}{i\sqrt{2}}, \\
T_{\pm 2}^{(2)} &= U_{\pm 1}V_{\pm 1}, \\
T_{\pm 1}^{(2)} &= \frac{U_{\pm 1}V_0 + U_0V_{\pm 1}}{\sqrt{2}}, \\
T_0^{(2)} &= \frac{U_{+1}V_{-1} + 2U_0V_0 + U_{-1}V_{+1}}{\sqrt{6}}
\end{aligned}$$

we can write

$$\begin{aligned}
&\langle \alpha', jm|\boldsymbol{J}\cdot\boldsymbol{V}|\alpha, jm\rangle \\
&\quad = \langle \alpha', jm|(J_0V_0 - J_{+1}V_{-1} - J_{-1}V_{+1})|\alpha, jm\rangle \\
&\quad = m\hbar\langle \alpha', jm|V_0|\alpha, jm\rangle \\
&\quad + \frac{\hbar}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\langle \alpha', j, m-1|V_{-1}|\alpha, jm\rangle \\
&\quad - \frac{\hbar}{\sqrt{2}}\sqrt{(j-m)(j+m+1)}\langle \alpha', j, m+1|V_{+1}|\alpha, jm\rangle \\
&\quad = c_{jm}\langle \alpha' j\|\boldsymbol{V}\|\alpha j\rangle,
\end{aligned}$$

where, according to the Wigner-Eckart theorem the coefficient c_{jm} does not depend on α, α' or $\boldsymbol{V}$.
The coefficient c_{jm} does not depend either on the quantum number m, because $\boldsymbol{J}\cdot\boldsymbol{V}$ is a scalar operator, so we can write it briefly as c_j. Because c_j does not depend on the operator $\boldsymbol{V}$ the above equation is valid also when $\boldsymbol{V}\to\boldsymbol{J}$ and $\alpha'\to\alpha$, or

$$\langle \alpha, jm|\boldsymbol{J}^2|\alpha, jm\rangle = \hbar^2 j(j+1) = c_j\langle \alpha j\|\boldsymbol{J}\|\alpha j\rangle.$$

If we now apply the Wigner-Eckart theorem to the operators V_q and J_q we get

$$\frac{\langle \alpha', jm'|V_q|\alpha, jm\rangle}{\langle \alpha, jm'|J_q|\alpha, jm\rangle} = \frac{\langle \alpha' j\|\boldsymbol{V}\|\alpha j\rangle}{\langle \alpha j\|\boldsymbol{J}\|\alpha j\rangle}.$$

for the ratios of the matrix elements. On the other hand, the right hand side of this equation is

$$\frac{\langle \alpha', jm|\boldsymbol{J}\cdot\boldsymbol{V}|\alpha, jm\rangle}{\langle \alpha, jm|\boldsymbol{J}^2|\alpha, jm\rangle},$$

so

$$\langle \alpha', jm'|V_q|\alpha, jm\rangle = \frac{\langle \alpha', jm|\boldsymbol{J}\cdot\boldsymbol{V}|\alpha, jm\rangle}{\hbar^2 j(j+1)}\langle jm'|J_q|jm\rangle \quad \blacksquare$$

Generalizing one can show that the reduced matrix elements of the irreducible product $T_q^{(k)}$ of two tensor operators, $X_{q_1}^{(k_1)}$ and $Z_{q_2}^{(k_2)}$, satisfy

$$\begin{aligned}
&\langle \alpha' j'||T^{(k)}\|\alpha j\rangle \\
&\quad = \sqrt{2k+1}(-1)^{k+j+j'}\sum_{\alpha''}\sum_{j''}\left\{\begin{array}{ccc} k_1 & k_2 & k \\ j & j' & j'' \end{array}\right\} \\
&\qquad \times\langle \alpha' j'\|X^{(k_1)}\|\alpha'' j''\rangle\langle \alpha'' j''|Z^{(k_2)}\|\alpha j\rangle.
\end{aligned}$$