Time dependent potentials

We have solved the problem

$$H_0|n\rangle = E_n|n\rangle$$

completely (here $|n\rangle \equiv |n^{(0)}\rangle$) and want to solve the eigenstates of the Hamiltonian

$$H = H_0 + V(t).$$

Since the Hamiltonian depends on time we have

$$\mathcal{U} \neq e^{-iHt/\hbar},$$

so a system in a stationary state $|i\rangle$ can in the course of time get components also in other stationary states.

Pictures of the time evolution

At the moment $t = 0$ let the system be in the state

$$|\alpha\rangle = \sum_n c_n(0)|n\rangle$$

and at the moment t in the state

$$|\alpha, t_0 = 0; t\rangle = \sum_n c_n(t)e^{-iE_nt/\hbar}|n\rangle.$$

Note The time dependence of the coefficients $c_n(t)$ is due only to the potential $V(t)$. The effect of the Hamiltonian H_0 is in the phase factors $e^{-iE_nt/\hbar}$. Our task is to evaluate the time-evolution of $c_n(t)$, i.e. to find the probability $|c_n(t)|^2$ that the system is found in state $|n\rangle$ at time t.

Schrödinger's picture

The evolution of the state vectors is governed by the time evolution operator:

$$|\alpha, t_0 = 0; t\rangle_S = \mathcal{U}(t)|\alpha, t_0 = 0\rangle.$$

Heisenberg's picture

The state vectors remain constant, i.e.

$$|\alpha, t_0 = 0, t\rangle_H = |\alpha, t_0 = 0\rangle.$$

On the other hand, the operators depend on time. We can go from the time independent operators of the Schrödinger picture to the operators of the Heisenberg picture via the transformation

$$A_H(t) = \mathcal{U}^\dagger(t)A_S\mathcal{U}(t).$$

If the Hamiltonian does not depend on time then

$$H_H(t) = \mathcal{U}^\dagger(t)H\mathcal{U}(t) = H$$

and

$$\frac{dA_H}{dt} = \frac{1}{i\hbar}[A_H, H_H] = \frac{1}{i\hbar}[A_H, H].$$

Interaction picture
The state vectors depend on time as

$$|\alpha, t_0; t\rangle_I \equiv e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S.$$

At the moment $t = 0$ we have obviously

$$|\rangle_S = |\rangle_I = |\rangle_H.$$

The interaction picture observables A_I are defined so that

$$A_I \equiv e^{iH_0t/\hbar}A_S e^{-iH_0t/\hbar}.$$

In particular we have

$$V_I = e^{iH_0t/\hbar}V e^{-iH_0t/\hbar}$$

in which $V = V_S(t)$.
We see that the equation governing the time dependence of the interaction picture state vectors is

$$\begin{aligned}
&i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I \\
&\quad= i\hbar\frac{\partial}{\partial t}\left(e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S\right) \\
&\quad= -H_0 e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S \\
&\qquad+e^{iH_0t/\hbar}(H_0 + V)|\alpha, t_0; t\rangle_S \\
&\quad= e^{iH_0t/\hbar}V e^{-iH_0t/\hbar}e^{iH_0t/\hbar}|\alpha, t_0; t\rangle_S,
\end{aligned}$$

where we inserted $i\hbar(\partial/\partial_t)|\alpha, t\rangle = H|\alpha, t\rangle$ and so

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = V_I|\alpha, t_0; t\rangle_I.$$

If now A_S does not depend on time we can derive

$$\frac{dA_I}{dt} = \frac{1}{i\hbar}[A_I, H_0],$$

which in turn resembles the Heisenberg equation of motion provided that in the latter we substitute for the total Hamiltonian H the stationary operator H_0.

	Heisenberg	Interaction	Schrödinger
State	No change	V_I	H
Operator	H	H_0	No change

Different time-evolution pictures;
the way the state kets and observables behave under time-evolution.

We see that interaction picture is something between the two extremes (Heisenberg and Schrödinger).
We expand state vectors in the base $\{|n\rangle\}$:

$$|\alpha, t_0; t\rangle_I = \sum_n c_n(t)|n\rangle.$$

If now $t_0 = 0$ so multiplying the previous expansion

$$|\alpha, t_0 = 0; t\rangle = \sum_n c_n(t)e^{-iE_n t/\hbar}|n\rangle$$

on both sides by the operator $e^{iH_0 t/\hbar}$ we get

$$|\alpha, t_0 = 0; t\rangle_I = \sum_n c_n(t)|n\rangle,$$

i.e. the coefficients c_n are equal. We just derived the equation

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_0; t\rangle_I = V_I|\alpha, t_0; t\rangle_I.$$

From this we get

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\langle n|\alpha, t_0; t\rangle_I &= \langle n|V_I|\alpha, t_0; t\rangle_I \\ &= \sum_m \langle n|V_I|m\rangle\langle m|\alpha, t_0; t\rangle_I. \end{aligned}$$

The matrix elements of the operator V_I are

$$\begin{aligned} \langle n|V_I|m\rangle &= \langle n|e^{iH_0 t/\hbar}V e^{-iH_0 t/\hbar}|m\rangle \\ &= V_{nm}(t)e^{i(E_n - E_m)t/\hbar} \end{aligned}$$

.
Because we furthermore have

$$c_n(t) = \langle n|\alpha, t_0; t\rangle_I,$$

we can write the equation governing the time dependence of the superposition coefficients as

$$i\hbar\frac{d}{dt}c_n(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t),$$

where

$$\omega_{nm} \equiv \frac{E_n - E_m}{\hbar} = -\omega_{mn}.$$

This system of differential equations can be written explicitely in the matrix form

$$i\hbar \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12}e^{i\omega_{12}t} & & \cdots \\ V_{21}e^{i\omega_{21}t} & V_{22} & & \cdots \\ & & V_{33} & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}.$$

In general, we need perturbative methods to solve these equations. Those few cases we can solve exactly are heavily implemented.

Example Two state systems with sinusoidal oscillating potential.

Suppose that

$$H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| \quad (E_1 < E_2)$$

and that the time dependent potential is like

$$V(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1| \quad \gamma, \omega \in \Re_+.$$

The matrix elements V_{nm} are now

$$\begin{aligned} V_{12} &= V_{21}^* &= \gamma e^{i\omega t} \\ V_{11} &= V_{22} &= 0, \end{aligned}$$

so transitions between the states $|1\rangle$ and $|2\rangle$ are possible. The system of differential equations to be solved is

$$\begin{aligned} i\hbar\dot{c}_1 &= \gamma e^{i\omega t} e^{i\omega_{12}t} c_2 \\ i\hbar\dot{c}_2 &= \gamma e^{-i\omega t} e^{i\omega_{21}t} c_1, \end{aligned}$$

where

$$\omega_{21} = -\omega_{12} = \frac{E_2 - E_1}{\hbar}.$$

We can see that the (exact) solution satisfying the initial conditions

$$c_1(0) = 1, \quad c_2(0) = 0$$

is

$$\begin{aligned} |c_2(t)|^2 &= \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \\ &\qquad \times \sin^2\left\{ \left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_{21})^2}{4} \right]^{1/2} t \right\} \\ |c_1(t)|^2 &= 1 - |c_2(t)|^2. \end{aligned}$$

The system oscillates between the states $|1\rangle$ and $|2\rangle$ with the angular velocity

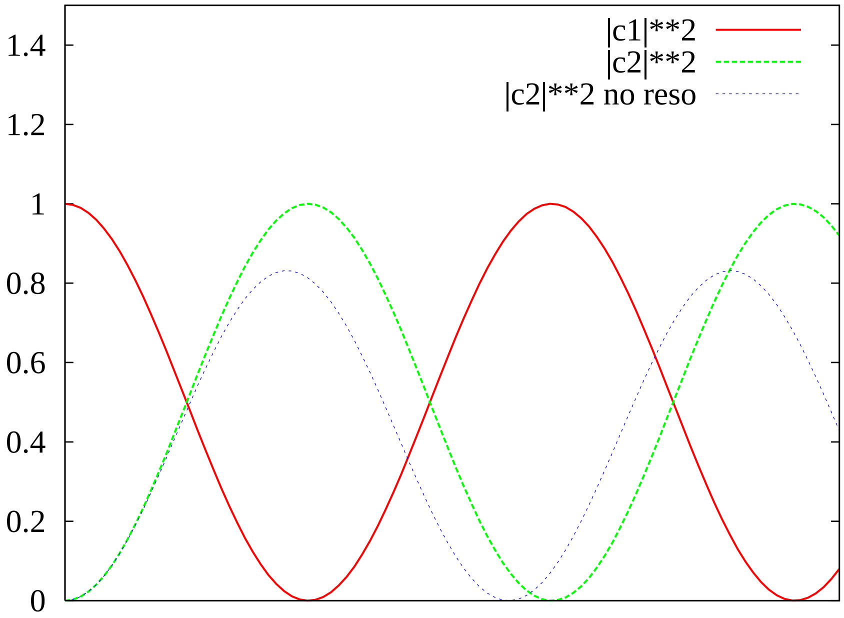
$$\Omega = \sqrt{\left(\frac{\gamma^2}{\hbar^2}\right) + \frac{(\omega - \omega_{21})^2}{4}}.$$

The probability to find the system in the higher energy state is given by

$$|c_2(t)|^2 = \frac{a^2}{\Omega^2} \sin^2(\Omega t).$$

The amplitude of the oscillations is at its maximum, i.e. we are in a resonance, when

$$\omega \approx \omega_{21} = \frac{E_2 - E_1}{\hbar}.$$



So, there is absorbtion-emission cycle of energy due to the V, the system is under control if we can control the V.

Example Spin magnetic resonance.

We put a spin $\frac{1}{2}$ particle (e.g. electron) into
time independent magnetic field parallel to the z axis,
time dependent magnetic field rotating in the xy plane, i.e.

$$\boldsymbol{B} = B_0 \hat{\boldsymbol{z}} + B_1(\hat{\boldsymbol{x}} \cos \omega t + \hat{\boldsymbol{y}} \sin \omega t)$$

where the fields B_0 and B_1 are constant. Since the magnetic moment of the electron is

$$\boldsymbol{\mu} = \frac{e}{m_e c} \boldsymbol{S} \quad \text{which acts like} \quad \boldsymbol{\mu} \cdot \mathbf{B},$$

the Hamiltonian is the sum of the terms

$$
\begin{aligned}
H_0 &= -\left(\frac{e\hbar B_0}{2m_e c}\right)\left(|S_z;\uparrow\rangle\langle S_z;\uparrow| - |S_z;\downarrow\rangle\langle S_z;\downarrow|\right) \\
V(t) &= -\left(\frac{e\hbar B_1}{2m_e c}\right) \\
&\quad \times\big[\cos\omega t(|S_z;\uparrow\rangle\langle S_z;\downarrow| + |S_z;\downarrow\rangle\langle S_z;\uparrow|) \\
&\quad + \sin\omega t(-i|S_z;\uparrow\rangle\langle S_z;\downarrow| + i|S_z;\downarrow\rangle\langle S_z;\uparrow|)\big].
\end{aligned}
$$

Since $e < 0$, then

$$
E_2 = E_\uparrow = \frac{|e|\hbar B_0}{2m_e c} > E_1 = E_\downarrow = -\frac{|e|\hbar B_0}{2m_e c}.
$$

We can thus identify in the above treated two stated system the quantities:

$$
\begin{array}{rcl}
|S_z;\uparrow\rangle & \mapsto & |2\rangle \quad \text{(higher state)} \\
|S_z;\downarrow\rangle & \mapsto & |1\rangle \quad \text{(lower state)} \\
\dfrac{|e|B_0}{m_e c} & \mapsto & \omega_{21} \\
-\dfrac{e\hbar B_1}{2m_e c} & \mapsto & \gamma, \quad \omega \mapsto \omega.
\end{array}
$$

Comparing with our earlier discussion on the spin precession we see that

- if $B_1 = 0$ and $B_0 \neq 0$, the the expectation value $\langle S_{x,y}\rangle$ rotates in the course of time counterclockwise but the probabilities $|c_1|^2$ and $|c_2|^2$ remain still constant.
- if $B_1 \neq 0$, the coefficients $|c_1|^2$ and $|c_2|^2$ are functions of time.

When the resonance condition

$$
\omega \approx \omega_{21}
$$

holds the probability for the spin flips

$$
|S_z;\uparrow\rangle \longleftrightarrow |S_z;\downarrow\rangle
$$

is very high.

Because experimental production of rotating magnetic fields is difficult it is common to use a field oscillating for example along the x axis. This can be divided into components rotating counterclockwise and clockwise:

$$
\begin{aligned}
&2B_1\hat{\boldsymbol{x}}\cos\omega t \\
&\qquad = B_1(\hat{\boldsymbol{x}}\cos\omega t + \hat{\boldsymbol{y}}\sin\omega t) \\
&\qquad\quad + B_1(\hat{\boldsymbol{x}}\cos\omega t - \hat{\boldsymbol{y}}\sin\omega t).
\end{aligned}
$$

In experiments one usually has

$$
\frac{B_1}{B_0} \ll 1,
$$

so

$$\frac{\gamma}{\hbar} = \frac{|e|B_1}{2m_e c} = \frac{|e|B_0}{m_e c}\frac{B_1}{2B_0} = \omega_{21}\frac{B_1}{2B_0} \ll \omega_{21}.$$

If now the component rotating counterclockwise triggers the resonance condition

$$\omega \approx \omega_{21},$$

the the transition probability due to this component is

$$\begin{aligned} |c_R(t)|^2 &= \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \\ &\quad \times \sin^2\left\{\left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega-\omega_{21})^2}{4}\right]^{1/2} t\right\} \\ &\approx \sin^2\left(\frac{\gamma}{\hbar}t\right). \end{aligned}$$

The clockwise rotating component,

$$\omega = -\omega_{21},$$

contributes

$$\begin{aligned} |c_L(t)|^2 &\approx \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + \omega_{21}^2} \\ &\quad \times \sin^2\left\{\left[\frac{\gamma^2}{\hbar^2} + \omega_{21}^2\right]^{1/2} t\right\} \\ &\ll |c_R(t)|^2. \end{aligned}$$

Thus,

- contribution from $|c_L(t)|^2$ is very small in magnitude and it oscillates very rapidly
- instead of rotating magnetic field we can apply oscillating field in x-direction, which is much easier to control
- by varying the frequency of the oscillating field, high precision measurements of magnetic moments $\boldsymbol{\mu}$ become possible
- Implementation of this resonance phenomenology on two-level systems has produced (only!) 4 Nobel prices

On top of SMR treated above, other applications are,
NMR, any atomic nucleus with odd number of nucleons (n, p, both fermions), possesses permanent magnetic moment
MASER, some molecules, ammonia NH_3 for instance, have 2 parity states close to each other. In external electric field these separate and

filtering/collecting the higher energy (anti-symmetric) states into a
"container" that is in external magnetic field, we get oscillating potential that gains the energy of the excited molecules.
LASER as an application of optical pumping
...up to you, more to come...