Identical particles

Treatment of identical particles differs dramatically among classical and quantum mechanics
In classical case, we can label the different particles as we wish and keep track on their positions and trajectories.
In quantum mechanics, identical particles are indistinguishable

- We can only label the quantum states according to the mutually commuting observables, adding extra labels is not possible
- We can not determine the position/trajectory without disturbing the state

Let us consider 2 particle state characterized by kets $|\alpha\rangle$ and $|\beta\rangle$ both of which represent collectively a complete set of observables.
The joint state reads $|12\rangle = |\alpha\rangle|\beta\rangle$.
Now the 2 particles could be in state $|12\rangle = |\beta\rangle|\alpha\rangle$.
$|\alpha\rangle|\beta\rangle$ and $|\beta\rangle|\alpha\rangle$ are 2 distinct states orhogonal to each other.
If we would now make a measurement to obtain eigenvalues of observables corresponding to α and β, both $|\alpha\rangle|\beta\rangle$ and $|\beta\rangle|\alpha\rangle$ as well as any linear combination $c_1|\alpha\rangle|\beta\rangle + c_2|\beta\rangle|\alpha\rangle$ would give the same eigenvalue, a quantum mechanical feature of quantum states composed of identical particles called *exchange degeneracy*.
Thus, unlike discussed before, a complete set of observables does not determine the state ket uniquely.
Let us define the permutation operator:

$$\begin{aligned} & P_{ij} \quad |1\rangle|2\rangle \dots |i\rangle \dots |j\rangle \dots |n\rangle \\ & = \quad |1\rangle|2\rangle \dots |j\rangle \dots |i\rangle \dots |n\rangle \end{aligned}$$

i.e. permutation operator exchanges 2 particles.
Quite obviously $P_{ij} = P_{ji}$ and $P_{ij}^2 = 1$.
Let $|a'\rangle$ and $|a''\rangle$ be eigenstates of observable A such that

$$A_1|a'\rangle|a''\rangle = a'|a'\rangle|a''\rangle \quad \text{and} \quad A_2|a'\rangle|a''\rangle = a''|a'\rangle|a''\rangle$$

in which 1 and 2 refer to the different particles and A_1 and A_2 the observables for those particles.
Now

$$\begin{aligned} P_{12}A_1|a'\rangle|a''\rangle &= a'P_{12}|a'\rangle|a''\rangle = a'|a''\rangle|a'\rangle \\ &= P_{12}A_1P_{12}^{-1}P_{12}|a'\rangle|a''\rangle \\ &= P_{12}A_1P_{12}^{-1}|a''\rangle|a'\rangle \end{aligned}$$

This can be only if $A_2 = P_{12}A_1P_{12}^{-1}$
So, P_{12} switches the indeces of observables as well.

Actually, any operator corresponding to a physical observable must be symmetrical in the change of 2 identical particles. From this follows for example that for N (identical) particle state

$$H = \sum_{i=1}^{N} \left(\frac{\boldsymbol{p_i}^2}{2m_i} + \sum_{j \neq i} V_{pair}(|\boldsymbol{x_i} - \boldsymbol{x_j}|) + V_{ext}(\boldsymbol{x_i}) \right)$$

and in particular

$$P_{ij} H P_{ij}^{-1} = H$$

Thus, P_{ij} is a constant of motion with eigenvalues ± 1 and so if the state is symmetric or antisymmetric, it will stay like that for ever.
Notice

$$[P_{ij}, P_{kl}] \neq 0$$

Example Let us consider 3 identical particles. Now there is 3!=6 kets of form $|a\rangle|b\rangle|c\rangle$. However, there is only 2 linear combinations that are totally symmetric or totally antisymmetric

$$\begin{aligned} |abc\rangle = \quad & \tfrac{1}{\sqrt{6}} \quad \Big(|a\rangle|b\rangle|c\rangle + |b\rangle|c\rangle|a\rangle + |c\rangle|a\rangle|b\rangle \\ & \pm \quad |b\rangle|a\rangle|c\rangle \pm |c\rangle|b\rangle|a\rangle \pm |a\rangle|c\rangle|b\rangle \Big) \end{aligned}$$

These are simultaneous eigenkets of P_{12}, P_{23} and P_{13}. Thus there 4 linear combinations that are neither symmetric nor antisymmetric.
Above $|a\rangle$, $|b\rangle$ and $|c\rangle$ are different states. In general, the normalization factor for a N identical particle state of whom N_a, N_b,...N_n are in states $|a\rangle$, $|b\rangle$...$|n\rangle$ reads

$$\sqrt{\frac{N_a! N_b! \ldots N_n!}{N!}}$$

It is an empirical fact that any system consisting of N identical particles is either totally symmetrical or antisymmetrical. No exceptions to this rule have been observed.
Thus, there appears to be 2 different kind of particles in nature. These are called bosons and fermions such that in the interchange of 2 identical particles in the N particle system

$$\begin{aligned} P_{ij}|\text{N identical bosons}\rangle &= +|\text{N identical bosons}\rangle \\ P_{ij}|\text{N identical fermions}\rangle &= -|\text{N identical fermions}\rangle \end{aligned}$$

Moreover, at this point we have to postulate

Particles with integer spin are bosons
Particles with $\frac{1}{2}$ integer spin are fermions
In the framework of relativistic QM it can be shown that
$\frac{1}{2}$ integer spin particles cannot be bosons and
integer spin particles cannot be fermions
This seems to be an exact law of nature without exceptions
Note Fermions obey Pauli exclusion principle, because any $|a\rangle|a\rangle$ is necessarily symmetrical.
Note Even composites of fermions are bosons.
The collective properties of these 2 types of particles are dramatically different.
Let us consider 2 particles that can occupy states $|a\rangle$ and $|b\rangle$.
For fermions there is only 1 possible state

$$\frac{1}{\sqrt{2}}(|a\rangle|b\rangle - |b\rangle|a\rangle)$$

For bosons, there are 3 possible states

$$|a\rangle|a\rangle \quad ; \quad |b\rangle|b\rangle \quad ; \quad \frac{1}{\sqrt{2}}(|a\rangle|b\rangle + |b\rangle|a\rangle)$$

For classical particles there would be 4 independent states

$$|a\rangle|a\rangle \quad ; \quad |b\rangle|b\rangle \quad ; \quad |a\rangle|b\rangle \quad ; \quad |b\rangle|a\rangle$$

Thus, both particles accopy the same state:
fermions in 0/1 cases
classical particles in 2/4 cases
bosons in 2/3 cases
At lectures, we will consider simple examples.
More general formalism to deal with many fermion systems can be found from EXTRAS/moi moi many_fermion_systems.ps
Also, handy way to keep track on permutations: Young tableau (googleit or see Sakurai)