Relativistic quantum mechanics

Non-relativistic QM (Dirac): "The underlying laws necessary for the mathematical theory of a large part of physics and the whole chemistry are completely known"
Still, missing relativistic part:
QM at relativistic speeds and
particle creation/annihilation, i.e. beta decay for example (antiparticles in general)
also "details" like fine structure of Hydrogen.
$\rightarrow$ necessity to introduce the concept of a field

Classical fields

We suppose that the Lagrange function

$$L = L(q_i, \dot{q}_i) = T - V$$

of classical mechanics does not depend explicitely on time. From the Hamilton variation principle

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt = 0, \quad \delta q_i(t)\big|_{t=t_{1,2}} = 0$$

one can derive the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.$$

The Hamiltonian function of the Hamiltonian mechanics is

$$H = \sum_i p_i \dot{q}_i - L,$$

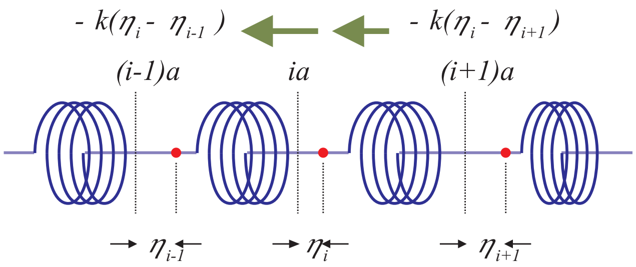
where the canonically conjugated momentum p_i of q_i is

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

The equations of motion are now

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}. \end{aligned}$$

Many body system



We consider N identical particles coupled to each other by identical parallel springs. The Lagrangian of the system is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}\sum_i^N \left[m\dot{\eta}_i^2 - k(\eta_{i+1}-\eta_i)^2\right], \end{aligned}$$

when η_i is the deviation of the particle i from its equilibrium position ia. We write this as

$$\begin{aligned} L &= \frac{1}{2}\sum_i^N \left[m\dot{\eta}_i^2 - k(\eta_{i+1}-\eta_i)^2\right] \\ &= \sum_i^N a\frac{1}{2}\left[\frac{m}{a}\dot{\eta}_i^2 - ka\left(\frac{\eta_{i+1}-\eta_i}{a}\right)^2\right] \\ &= \sum_i^N a\mathcal{L}_i. \end{aligned}$$

Here $\mathcal{L}_i$ is the linear Lagrangian density.
We go to continuum by substituting the limits

$$\begin{aligned} a &\rightarrow dx \\ \frac{m}{a} &\rightarrow \mu = \text{linear mass density} \\ \frac{\eta_{i+1}-\eta_i}{a} &\rightarrow \frac{\partial\eta}{\partial x} \\ ka &\rightarrow Y = \text{Young's modulus}. \end{aligned}$$

Now

$$L = \int \mathcal{L}\, dx,$$

where

$$\mathcal{L} = \frac{1}{2}\left[\mu\dot{\eta}^2 - Y\left(\frac{\partial\eta}{\partial x}\right)^2\right].$$

In the continuous case the Hamiltonian variation principle takes the form

$$\delta\int_{t_1}^{t_2} L\, dt$$

$$
\begin{aligned}
&= \delta \int_{t_1}^{t_2} dt \int dx \, \mathcal{L}\left(\eta, \dot{\eta}, \frac{\partial \eta}{\partial x}\right) \\
&= \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial \eta} \delta\eta + \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)} \delta\left(\frac{\partial \eta}{\partial x}\right) \right. \\
&\quad \left. + \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)} \delta\left(\frac{\partial \eta}{\partial t}\right) \right\} \\
&= \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial \eta} \delta\eta + \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)} \frac{\partial}{\partial x}(\delta\eta) \right. \\
&\quad \left. + \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)} \frac{\partial}{\partial t}(\delta\eta) \right\} \\
&= \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial \eta} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)}\right) \right. \\
&\quad \left. - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)}\right) \right\} \delta\eta.
\end{aligned}
$$

Last step: intergrate by parts and remember that $\delta\eta$ must vanish at the end points of the integral.
To get the variation to vanish for all $\delta\eta$ one must satisfy the Euler-Lagrange equation

$$
\frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)} + \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)} - \frac{\partial \mathcal{L}}{\partial \eta} = 0.
$$

When

$$
\mathcal{L} = \frac{1}{2}\left[\mu \dot{\eta}^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2\right],
$$

then

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \eta} &= 0 \\
\frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)} &= -Y \frac{\partial}{\partial x} \frac{\partial \eta}{\partial x} = -Y \frac{\partial^2 \eta}{\partial x^2} \\
\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)} &= \mu \frac{\partial^2 \eta}{\partial t^2}.
\end{aligned}
$$

Substituting into the Euler-Lagrange equation we get

$$
Y \frac{\partial^2 \eta}{\partial x^2} - \mu \frac{\partial^2 \eta}{\partial t^2} = 0,
$$

which describes a wave progating in one dimension with the velocity $\sqrt{Y/\mu}$.
We define the canonically conjugated momentum

$$
\pi = \frac{\partial \mathcal{L}}{\partial \dot{\eta}}
$$

and the Hamiltonian density

$$\mathcal{H} = \dot{\eta}\pi - \mathcal{L}.$$

Now

$$\pi = \mu\dot{\eta},$$

so

$$\begin{aligned}\mathcal{H} &= \mu\dot{\eta}^2 - \frac{1}{2}\left[\mu\dot{\eta}^2 - Y\left(\frac{\partial\eta}{\partial x}\right)^2\right] \\ &= \frac{1}{2}\mu\dot{\eta}^2 + \frac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2.\end{aligned}$$

which is the sum of kinetic and potential energy densities.
The Lagrangian formalism generalizes easily to three dimensions. The Euler-Lagrange equation takes then the form

$$\sum_{k=1}^{3}\frac{\partial}{\partial x_k}\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_k)} + \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial t)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

Covariant formulation
In *special relativity*, metrics do not play such a crucial role as in general relativity (due to the flat space-time).
What will follow may seem sloppy, even incorrect for the ones familiar with general relativity. Here, metrics is taken into account by introducing imaginary time component.
And we do not introduce metric tensor at all...
Einstein summation convention is used for greek indeces.
This leads to

$$a^\mu \equiv a_\mu \quad \rightarrow \quad a^\mu b^\mu \equiv a_\mu b_\mu \equiv a_\mu b^\mu$$

We employ the metrics where the components of a four-vector b_μ are

$$b_\mu = (b_1, b_2, b_3, b_4) = (\boldsymbol{b}, ib_0).$$

In particular the coordinate four-vector is

$$x_\mu = (x_1, x_2, x_3, x_4) = (\boldsymbol{x}, ict).$$

Under Lorentz transformations the coordinate vector transforms according to the equation

$$x'_\mu = a_{\mu\nu}x_\nu.$$

The coefficients of the Lorentz transformation satisfy the orthogonality condition

$$a_{\mu\nu}a_{\mu\lambda} = \delta_{\nu\lambda}, \quad (a^{-1})_{\mu\nu} = a_{\nu\mu},$$

so that

$$x_\mu = (a^{-1})_{\mu\nu} x'_\nu = a_{\nu\mu} x'_\nu.$$

We define the *four-vector* so that under Lorentz transformations it transforms like x_μ.
Now

$$\frac{\partial}{\partial x'_\mu} = \frac{\partial x_\nu}{\partial x'_\mu} \frac{\partial}{\partial x_\nu} = a_{\mu\nu} \frac{\partial}{\partial x_\nu},$$

so the four-gradient $\partial / \partial x_\mu$ is a four-vector.
The scalar product of the four-vectors b and c,

$$b_\mu c_\mu = \sum_{j=1}^{3} b_j c_j + b_4 c_4 = \boldsymbol{b} \cdot \boldsymbol{c} - b_0 c_0,$$

is invariant under Lorentz transformations.
Using the four-vector notation the Euler-Lagrange equation can be written into the compact form

$$\frac{\partial}{\partial x_\mu} \left[\frac{\partial \mathcal{L}}{\partial(\partial \phi / \partial x_\mu)} \right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

We see that the field equation derived from the Lagrangian density $\mathcal{L}$ is covariant provided that the Lagrangian density $\mathcal{L}$ itself is relativistically scalar (invariant under Lorentz transformations).

Klein-Gordon's equation

We consider the scalar field $\phi(x)$ which, according to its definition, behaves under Lorentz transformation like

$$\phi'(x') = \phi(x).$$

Now

$$\mathcal{L} = \mathcal{L}(\phi, \partial \phi / \partial x_\mu).$$

Since we want

- the Lagrangian density to be invariant under Lorentz transformations
- a linear wave equation,

the Lagrangian density can contain only the terms

$$\phi^2 \text{ and } \frac{\partial \phi}{\partial x_\mu} \frac{\partial \phi}{\partial x_\mu}.$$

One possible form for the Lagrangian density is

$$\mathcal{L} = -\frac{1}{2} \left(\frac{\partial \phi}{\partial x_\mu} \frac{\partial \phi}{\partial x_\mu} + \mu^2 \phi^2 \right).$$

Substituting this into the Euler-Lagrange equation

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x_\mu)}\right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0,$$

we get

$$-\frac{\partial}{\partial x_\mu}\left(\frac{\partial \phi}{\partial x_\mu}\right) + \mu^2\phi = 0.$$

If we employ the notation

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2},$$

we end up with the *Klein-Gordon* equation

$$\Box\phi - \mu^2\phi = 0.$$

This is the relativistic correspondence for the Schrödinger equation of a spinless free particle.

Heuristic derivation
We substitute into the relativistic energy-momentum relation

$$E^2 - |\boldsymbol{p}|^2c^2 = m^2c^4$$

the operators

$$E \mapsto i\hbar\frac{\partial}{\partial t}, \quad p_k \mapsto -i\hbar\frac{\partial}{\partial x_k},$$

and get

$$\left(-\frac{\partial^2}{c^2\partial t^2} + \nabla^2 - \frac{m^2c^2}{\hbar^2}\right)\phi = 0.$$

When we set

$$\mu = \frac{mc}{\hbar}, \ [\mu] = \frac{1}{\text{length}},$$

we end up with the Klein-Gordon equation.
There are no sources in the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x_\mu} + \mu^2\phi^2\right)$$

so the solution describes a free field. We include the term

$$\mathcal{L}_{\text{int}} = -\phi\rho,$$

where ρ is the (usually position dependent) density of the source. The field equation is now

$$\Box\phi - \mu^2\phi = \rho.$$

When we choose (point-like source at origin)

$$\rho = G\delta(\boldsymbol{x})$$

and seek for a stationary solution we end up with the equation

$$(\nabla^2 - \mu^2)\phi = G\delta(\boldsymbol{x}) \quad (*).$$

We substitute ϕ using its Fourier transform

$$\phi(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tilde{\phi}(\boldsymbol{k}),$$

where

$$\tilde{\phi}(\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \phi(\boldsymbol{x}).$$

Multiplying both sides of (*) with $\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}/(2\pi)^{3/2}$ and integrating over d^3x we end up with the algebraic equation

$$(-\boldsymbol{k}^2 - \mu^2)\tilde{\phi}(\boldsymbol{k}) = \frac{G}{(2\pi)^{3/2}}$$

of the Fourier components. Its solution is

$$\tilde{\phi}(\boldsymbol{k}) = -\frac{G}{(2\pi)^{3/2}} \frac{1}{k^2 + \mu^2}.$$

Taking the Fourier transform we get the solution

$$\phi(\boldsymbol{x}) = -\frac{G}{4\pi} \frac{e^{-\mu r}}{r},$$

known as the *Yukawa potential.*

Before quark model and QCD, nucleon-nucleon interaction was considered to be mediated by pions.

A nucleon would be a source to the meson field (like electron to the foton field)

Let's suppose that the meson field of a nucleon at the point $\boldsymbol{x}_1$ satisfies the equations

$$(\nabla_2^2 - \mu^2)\phi = G\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2).$$

Its solution is thus the Yukawa potential.

$$\phi(\boldsymbol{x}_2) = -\frac{G}{4\pi} \frac{e^{-\mu|\boldsymbol{x}_2 - \boldsymbol{x}_1|}}{|\boldsymbol{x}_2 - \boldsymbol{x}_1|}.$$

Because the Hamiltonian density was

$$\mathcal{H} = \dot{\eta}\pi - \mathcal{L},$$

the Hamiltonian density of the interaction is

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$$

and the total interaction Hamiltonian

$$H_{\text{int}} = \int \mathcal{H}_{\text{int}}\, d^3x = \int \phi\rho\, d^3x.$$

We see that the interaction energy of nucleons located at the points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is

$$H_{\text{int}}^{(1,2)} = -\frac{G}{4\pi}\frac{e^{-\mu|\boldsymbol{x}_2-\boldsymbol{x}_1|}}{|\boldsymbol{x}_2-\boldsymbol{x}_1|}.$$

Note Unlike in the Coulomb case, this interaction is attractive (for both nucleons: n-n, n-p, p-p) and short ranged and thus grasps qualitatively some piece of the strong interactions.

Complex scalar fields

In the reality there are 3 mesons (π^+, π^0, π^-), with different charges but with (almost) equal masses, consistent with the theory. We expand our theory so that we consider two real fields, ϕ_1 and ϕ_2, for two particles with equal masses. From these we construct the complex fields

$$\begin{aligned}\phi &= \frac{\phi_1 + i\phi_2}{\sqrt{2}}\\ \phi^* &= \frac{\phi_1 - i\phi_2}{\sqrt{2}}.\end{aligned}$$

The Lagrangian density for the free fields can be written using either the complex or real fields:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\left(\frac{\partial\phi_1}{\partial x_\mu}\frac{\partial\phi_1}{\partial x_\mu} + \mu^2\phi_1^2\right) - \frac{1}{2}\left(\frac{\partial\phi_2}{\partial x_\mu}\frac{\partial\phi_2}{\partial x_\mu} + \mu^2\phi_2^2\right)\\ &= -\left(\frac{\partial\phi^*}{\partial x_\mu}\frac{\partial\phi}{\partial x_\mu} + \mu^2\phi^*\phi\right).\end{aligned}$$

Considering the fields ϕ and ϕ^* independent we get two Euler-Lagrange equations

$$\begin{aligned}\frac{\partial}{\partial x_\mu}\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_\mu)} - \frac{\partial\mathcal{L}}{\partial\phi} &= 0\\ \frac{\partial}{\partial x_\mu}\frac{\partial\mathcal{L}}{\partial(\partial\phi^*/\partial x_\mu)} - \frac{\partial\mathcal{L}}{\partial\phi^*} &= 0,\end{aligned}$$

which can be further written as two Klein-Gordon equations

$$\begin{aligned}\Box\phi^* - \mu^2\phi^* &= 0\\ \Box\phi - \mu^2\phi &= 0.\end{aligned}$$

These describe two free fields (of same masses) with opposite charges. Explanation will follow...
We define the *first order gauge transformation* so that the fields transform under it like

$$\begin{aligned} \phi' &= e^{i\lambda}\phi \\ \phi^{*\prime} &= e^{-i\lambda}\phi^*, \end{aligned}$$

when λ is a real parameter. Let λ be now an arbitrary, infinitesimally small, number. Then

$$\begin{aligned} \delta\phi &= i\lambda\phi \\ \delta\phi^* &= -i\lambda\phi^*. \end{aligned}$$

The Lagrangian density transforms then as

$$\begin{aligned} \delta\mathcal{L} &= \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_\mu)}\delta\left(\frac{\partial\phi}{\partial x_\mu}\right)\right] \\ &\quad + \left[\frac{\partial\mathcal{L}}{\partial\phi^*}\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial\phi^*/\partial x_\mu)}\delta\left(\frac{\partial\phi^*}{\partial x_\mu}\right)\right] \\ &= \left[\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_\mu)}\right)\right]\delta\phi \qquad (*) \\ &\quad + \left[\frac{\partial\mathcal{L}}{\partial\phi^*} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi^*/\partial x_\mu)}\right)\right]\delta\phi^* \quad (*) \\ &\quad + \frac{\partial}{\partial x_\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_\mu)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial\phi^*/\partial x_\mu)}\delta\phi^*\right] \\ &= -i\lambda\frac{\partial}{\partial x_\mu}\left(\frac{\partial\phi^*}{\partial x_\mu}\phi - \phi^*\frac{\partial\phi}{\partial x_\mu}\right). \end{aligned}$$

(*)s are zero according to Lagrange-Euler eq. and the rest follows from taking partial derivatives of

$$\mathcal{L} = -\left(\frac{\partial\phi^*}{\partial \S_\mu}\frac{\partial\phi}{\partial \S_\mu} + \mu^{\in}\phi^*\phi\right)$$

In a small neighborhood of the solutions ϕ and ϕ^* the Lagrangian density is invariant so we must have

$$\delta\mathcal{L} = 0.$$

Thus we get

$$\frac{\partial s_\mu}{\partial x_\mu} = 0,$$

where

$$s_\mu = i\left(\frac{\partial\phi^*}{\partial x_\mu}\phi - \phi^*\frac{\partial\phi}{\partial x_\mu}\right).$$

We see that

- a complex field ϕ is associated with a conserved four-vector density s_μ,
- if we exchange $\phi \longleftrightarrow \phi^*$, then $s_\mu \longleftrightarrow -s_\mu$.

We interpret this so that
s_μ is the charge current density
(any internal conserved quantum number),
ϕ carries the charge e,
ϕ^* carries the charge $-e$,
the previous real field corresponds to neutral mesons.
Notice the connection with our earlier symmetry considerations (Noether's theorem)