1. Consider a three dimensional ket space. If a certain set of orthornormal kets, say $|1\rangle$, $|2\rangle$ and $|3\rangle$ are used as the base kets, then the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

in which both a and b are real.

   a) Obviously A exhibits degenerate spectrum. Does B have degenerate spectrum as well?

   b) Show that A and B commute.

   c) Find a new (orthonormal) set of base kets which are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

2. Evaluate the uncertainty relation of x and p operators for a particle confined in an infinite potential well (between two unpenetrable walls).

Some help:
In this case the potential can be written: $V = 0$ for $0 < x < a$ while $V = \infty$ otherwise. From quantum mechanics I we remember that the wave function in such a potential well reads

$$\psi_\alpha(x') = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x'}{a}\right)$$

in which n refers to the nth excitation $n = 1$ being the ground state. Compare this with the uncertainty relation for harmonic oscillator.

3. Show that

   a)

$$\langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$$

   b)

$$\langle\beta|x|\alpha\rangle = \int \mathrm{d}p'\phi^*_\beta(p')\mathrm{i}\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$$

4. The concept of pure ensemble in a nutshell:
   In a pure ensemble every state is the same, thus the statistical mechanics of such an ensemble reduces to ordinary quantum mechanics.

   a) Show that the relation $\rho^2 = \rho$ holds for the density operator of a pure ensemble and thus $\mathrm{tr}(\rho^2)=1$.

   b) Show that the corresponding matrix representation is given by a matrix where one of the diagonal elements is 1 and the rest are zero. Is this always true?

5. Suppose we have a mixed ensemble of spin-$\frac{1}{2}$ particles with 1/3 of the ensemble in the state $|S_z;\uparrow\rangle$ and the rest in the state $|S_z;\downarrow\rangle$.

   a) Find a matrix representation for the density operator in the $\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$ basis as well as in the $\{|S_x;\uparrow\rangle, |S_x;\downarrow\rangle\}$ basis.

   b) Calculate the ensemble averages of S_x, S_y and S_z in this ensemble. Calculate the same averages in unpolarized ensemble (completely mixed ensemble) for which the density operator reads:

$$\rho = \frac{1}{2}\left(|S_z;\uparrow\rangle\langle S_z;\uparrow| + |S_z;\downarrow\rangle\langle S_z;\downarrow|\right).$$

   Comment your findings.

   c) In spherical coordinates the unit vector $\hat{\mathbf{n}}$ can be written

$$\hat{\mathbf{n}} = \sin\theta\cos\phi\ \hat{\imath} + \sin\theta\sin\phi\ \hat{\jmath} + \cos\theta\ \hat{k}$$

   Find $[\mathbf{S}\cdot\hat{\mathbf{n}}]$ for an arbitrary axis $\hat{\mathbf{n}}$ for the two ensembles considered above. Verbalize your results.