76322S ADVANCED QUANTUM MECHANICS Exercise 3 2007

1. Consider spin precession of electron in static uniform magnetic field in the z -direction and calculate the expectation values of spin at time t in y and z directions when the initial state of the system at t=0 is $|S_x;\uparrow\rangle = \frac{1}{\sqrt{2}}|S_z;\uparrow\rangle + \frac{1}{\sqrt{2}}|S_z;\downarrow\rangle$.
   Remember identities:
   $|S_y;\uparrow\rangle = \frac{1}{\sqrt{2}}|S_z;\uparrow\rangle + i\frac{1}{\sqrt{2}}|S_z;\downarrow\rangle$
   $|S_y;\downarrow\rangle = \frac{1}{\sqrt{2}}|S_z;\uparrow\rangle - i\frac{1}{\sqrt{2}}|S_z;\downarrow\rangle$.

2. Consider one-dimensional simple harmonic oscillator.

   a) Construct a linear combination of $|0\rangle$ (the ground state) and $|1\rangle$ (the 1st excited state) such that $\langle x\rangle$ is as large as possible.

   b) Let the system be initially in the state constructed above: $|\alpha, t_0\rangle$. What is the state vector $|\alpha, t_0, t\rangle$ at time t in the Schrödinger picture? Evaluate the expectation value $\langle x\rangle$ as a function of time in the Schrödinger and in the Heisenberg picture.

   c) Choose either Schrödinger or Heisenberg picture and evaluate $\langle(\Delta x)^2\rangle$.

3. Derive the formal solutions of the Schrödinger equation for time-evolution operator

$$i\hbar\frac{\partial}{\partial t}\mathcal{U}(t,t_0) = H\mathcal{U}(t,t_0)$$

   in the three cases considered during lectures:

   a) Hamiltonian is time-independent

   b) Hamiltonians at different time commute

   c) Hamiltonians at different times do not commute

4. Consider a physical system whose three-dimensional state space is spanned by the orthonormal basis formed by $|u_1\rangle$, $|u_2\rangle$ and $|u_3\rangle$. In this basis, the Hamiltonian H of the system and two observables A and B have matrix representations

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & b \end{pmatrix}$$

where all constants are positive and real. The initial state is

$$|\alpha, t_0 = 0\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

a) The energy of the system is measured at $t = 0$. What energies with which probabilities can be found? What are the mean energy and variance (dispersion) of energy?

b) If A is measured at $t = 0$, what values can be found and with which probabilities? What is the state vector immediately after the measurement?

c) What is $|\alpha, t_0, t\rangle$?

d) Calculate the expectation values of A and B as function of time. Can you comment something?

e) What results can be obtained if A is measured at time t. How about B? Again, try to verbalize your results.

5. Two observables A_1 and A_2 which do not involve time explicitly, are known NOT to commute,

$$[A_1, A_2] \neq 0.$$

However, both of them commute with the Hamiltonian

$$[A_1, H] = 0 \quad , \quad [A_2, H] = 0.$$

Show that the energy eigenstates are, in general, degenerate. Are there exceptions? For example, you can consider the central-force problem $H = \mathbf{p}^2/2m + V(r)$ with $A_1 = L_z$ and $A_2 = L_x$.