76322S ADVANCED QUANTUM MECHANICS Exercise 4 2007

1. Let us have an ensemble of spin-$\frac{1}{2}$ particles. You can think of for example electrons in Stern-Gerlach experiment.

   a) Consider a pure ensemble, i.e. a case where we have performed an S-G experiment and blocked the other spin component and let the other get through in such a way that we do not actually know which state we have selected. Instead, we know the expectation values of $\langle S_x \rangle$ and $\langle S_z \rangle$ as well as the sign of $\langle S_y \rangle$. How can we find the state vector corresponding our ensemble.

   b) Consider a mixed ensemble of spin-$\frac{1}{2}$ particles in which we do not know the relative fractions of spin up and down states in our ensemble but the ensemble averages $[S_x]$, $[S_y]$ and $[S_z]$ we do know. Find a 2×2 density matrix characterizing our ensemble.

2. An electron (i.e. spin $\frac{1}{2}$-particle) is known to be in an eigenstate of $\mathbf{S}\cdot\hat{\mathbf{n}}$ with eigenvalue $\frac{\hbar}{2}$. The unit vector $\hat{\mathbf{n}}$ is such that it lays in the xz-plane and makes an angle β with the positive z-axis.

   a) Let us measure S_x. What is the probability that the result is $\frac{\hbar}{2}$?

   b) Calculate the expectation value of S_x as a function of time.

   c) Check if your result makes sense in the special cases $\beta = 0$ and $\beta = \frac{\pi}{2}$

3. Consider an electron-positron system in an external uniform magnetic field in the z-direction. The Hamiltonian of the system can be written as

$$H = A\mathbf{S}^{(e^-)} \cdot \mathbf{S}^{(e^+)} + \frac{eB}{mc}\left(S_z^{(e^-)} - S_z^{(e^+)}\right)$$

in which the first term describes the electron-positron spin-spin coupling and the second term the coupling of the spins of electron and positron with the magnetic field. Let the system be in the state $|\alpha\rangle = \chi_\uparrow^{(e^-)}\chi_\downarrow^{(e^+)}$.

Is $|\alpha\rangle$ an eigenfunction of H?
If it is, what is the eigenvalue?
If not, what is the expectation value?

a) in the case there is no spin coupling ($A \rightarrow 0$)

b) in the case ($A \neq 0$) during electricity blackout (i.e. when $B \rightarrow 0$)

Notice that spinors for electron and positron are in different spaces and thus the operators for electron and positron act in different spaces as well.

4. Show that

$$\exp\left(\frac{-i\sigma \cdot \hat{\mathbf{n}}\phi}{2}\right) = \mathbf{1}\cos\frac{\phi}{2} - i\sigma \cdot \hat{\mathbf{n}}\sin\frac{\phi}{2}.$$

5. Consider the experiment that measured the gravity induced quantum interference. In this apparatus, thermal neutrons are directed through a christal via 2 separate paths. Obviously, some energy-momentum filtering of the thermal neutrons must be done so that approximately similar wavepackets are entering the christal. The wavelength of neutrons chosen was 1.445 Angström. Lets assume that the apparatus had a high performance filter and that it could choose the wavelength of the neutrons entering the system at the level of $\Delta\lambda/\lambda = 10^{-9}$. Estimate the length of the wavepacket in the direction of propagation. What do you think, what was the actual error in wave length.