1. Derive the Clebsch-Gordan symmetry relation

$$\langle j_1 j_2; m_1 m_2 | j_1 j_2; j_3 m_3\rangle = (-1)^{j_2+m_2}\sqrt{\frac{2j_3+1}{2j_1+1}}\langle j_2 j_3; -m_2 m_3 | j_2 j_3; j_1 m_1\rangle$$

2. A system is described by a wave function

$$\psi(x,y,z) = N(x+y+z)\mathrm{e}^{-r^2/\alpha^2},$$

   in which α is given and real and N the normalization constant. The observables $\mathbf{L^2}$ and L_z are measured. What are the possible results and the corresponding probabilities?
   Hint:A good starting point would be to decompose the wave function as spherical harmonics times the radial part.

3. On Spherical harmonics.

   a) Write xy, xz and (x^2-y^2) as components of a spherical (irreducible) tensor of rank 2.

   b) The quadrupole moment is the expectation value

   $$Q = e\langle \alpha, j, m=j|(3z^2-r^2)|\alpha, j, m=j\rangle.$$

   Evaluate

   $$e\langle \alpha, j, m'|(x^2-y^2)|\alpha, j, m=j\rangle$$

   ($m' = j, j-1, j-2, ...$) in terms of Q and Clebsch-Gordan coefficients.

4. A quantum mechanical state Ψ is known to be simultaneous eigen-state of two Hermitian operators A and B which anticommute

$$\{A,B\} = AB + BA = 0$$

   What can you say about the eigenvalues of A and B for the state Ψ? Illustrate your point using the parity (A) and momentum (B) operators.

5. Let $\mathcal{T}(\mathbf{d})$ be the translation operator with displacement $\mathbf{d}$, $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ the rotation operator and π the parity operator. Which, if any of the following pairs commute? Why?

   a) $\mathcal{T}(\mathbf{d})$ and $\mathcal{T}(\mathbf{d}')$ (different directions)

   b) $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ and $\mathcal{D}(\hat{\mathbf{n}}', \phi')$ ($\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ in different directions)

   c) $\mathcal{T}(\mathbf{d})$ and π

   d) $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ and π