1. Apply perturbation theory to a one-dimensional simple harmonic oscillator which is subjected to a linear perturbation

$$V = -\lambda x$$

in which λ is a real and known constant.

   a) Calculate the energy shift of the ground state up to the lowest non-vanishing order using (non-degenerate time-independent) perturbation theory

   b) Solve the problem exactly and compare the exact result with the result coming from perturbation theory.

Hint: We have seen before that in the Hamiltonian eigenbasis $H|n\rangle = (n + \frac{1}{2})\hbar w|n\rangle$ the position operator matrix elements read

$$\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2mw}}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1})$$

2. Consider a particle in a one-dimensional infinite potential well: $V = 0$ for $|x| < a$ while $V = \infty$ for $|x| > a$. Estimate ground state wave-function and corresponding ground state energy by implementing the variational method.
Hint: Physical requirements are that the wave function must vanish at $x = \pm a$ and that there are no "wiggles" (no waves) in the region $|x| < a$. These two requirements are satisfied by a parabola

$$\langle x|\overline{O}\rangle = a^2 - x^2$$

or more generally by a polynomial function of the degree $\lambda \leq 2$:

$$\langle x|\overline{O}\rangle = |a|^\lambda - |x|^\lambda .$$

Consider λ as a variational parameter and find the ground state.
Compare your result with the well-known exact result

$$\langle x|0\rangle = \frac{1}{\sqrt{a}}\cos\left(\frac{\pi x}{2a}\right) \quad ; \quad E_0 = \left(\frac{\hbar^2}{2m}\right)\left(\frac{\pi^2}{4a^2}\right).$$

3. Consider a particle in a 2-dimensional infinite potential well

$$\begin{aligned} V_0 &= 0 \quad \text{for} \quad 0 \leq x \leq L \quad \text{and} \quad 0 \leq y \leq L \\ V_0 &= \infty \quad \text{otherwise} \end{aligned}$$

Write out the energy eigenfunctions for the ground state and for the first excited state. Let us now apply a constant force to the system (inside the well), i.e. let us introduce a time-independent perturbation

$$\begin{aligned} V_1 &= \lambda xy \quad \text{for} \quad 0 \leq x \leq L \quad \text{and} \quad 0 \leq y \leq L \\ V_1 &= \infty \quad \text{otherwise} \end{aligned}$$

Use perturbation theory to obtain the zeroth order energy eigenfunctions and first order energy shifts both for the ground and for the first excited states. You might find

$$\begin{aligned} \int x \sin(ax) \sin(bx) dx &= \frac{1}{2}\left(\frac{\cos(ax-bx)}{(a-b)^2} + \frac{x\sin(ax-bx)}{a-b}\right) \\ &- \frac{1}{2}\left(\frac{\cos(ax+bx)}{(a+b)^2} + \frac{x\sin(ax+bx)}{a+b}\right) \end{aligned}$$

useful at some point of your calculation.

4. Consider a p orbital electron in an atom. The states are characterized by the kets $|n, l = 1, m = -1, 0, 1\rangle$. Again, spin is ignored and so the energy eigenvalues are degenerate. Let us subject the atom to a potential

$$V = \lambda(x^2 - y^2)$$

in which λ is real and pre-determined constant.

   a) Find the unperturbed energy eigenkets that diagonalize the perturbation. You do not have to evaluate the energy shifts in detail, just show that the degeneracy is completely removed.

   b) Since the perturbation V is invariant under time-reversal and there is no more degeneracy, we expect each of the energy eigenstates obtained above to go into themselves under time-reversal. Verify this by explicit calculation.